Which simple types have a unique inhabitant?

Gabriel Scherer, Didier Rémy

Gallium – INRIA

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- in a given type system (STLC with atoms, products and sums)
- modulo some program equivalence ($\beta\eta$)
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- in a given type system (STLC with atoms, products and sums)
- modulo some program equivalence ($\beta\eta$)

Motivation: a principal way to study code inference.

We can infer the instance declaration for the exception monad $A \mapsto A + E$:

$$X + E \rightarrow (X \rightarrow Y + E) \rightarrow Y + E$$
STLC with sums

\[
\begin{align*}
\Delta, x : A & \vdash t : B \\
\frac{}{\Delta \vdash \lambda x. t : A \to B} \\
\end{align*}
\]

\[
\begin{align*}
\Delta & \vdash t : A \to B \\
\Delta & \vdash u : A \\
\frac{}{\Delta \vdash t \ u : B} \\
\end{align*}
\]

\[
\begin{align*}
\Delta & \vdash t : A \\
\Delta & \vdash u : B \\
\frac{}{\Delta \vdash (t, u) : A \times B} \\
\end{align*}
\]

\[
\begin{align*}
\Delta & \vdash t : A_1 \times A_2 \\
\frac{}{\Delta \vdash \pi_i \ t : A_i} \\
\end{align*}
\]

\[
\begin{align*}
\Delta & \vdash t : A_i \\
\frac{}{\Delta \vdash \sigma_i \ t : A_1 + A_2} \\
\end{align*}
\]

\[
\begin{align*}
\Delta, x_1 : A_1 & \vdash u_1 : C \\
\Delta, x_2 : A_2 & \vdash u_2 : C \\
\frac{}{\Delta \vdash \delta(t, x_1. u_1, x_2. u_2) : C} \\
\end{align*}
\]
\[ (\lambda x. t) u \rightarrow_{\beta} t[u/x] \quad (t : A \rightarrow B) =_{\eta} \lambda x. t x \]

\[ \pi_i (t_1, t_2) \rightarrow_{\beta} t_i \quad (t : A \times B) =_{\eta} (\pi_1 t, \pi_2 t) \]

\[ \delta(\sigma_i t, x_1.u_1, x_2.u_2) \rightarrow_{\beta} u_i[t/x_i] \]

\[ \forall C[\Box], \quad C[t : A + B] =_{\eta} \delta(t, x.C[\sigma_1 x], x.C[\sigma_2 x]) \]

A primer on focusing
Sequent calculus

(Can be done in natural deduction, but less regular)

\[ \Delta \vdash A \quad \Delta, B \vdash C \]
\[ \frac{\Delta, A \rightarrow B \vdash C}{\Delta, A \rightarrow B} \]
\[ \Delta, A \vdash B \]
\[ \frac{\Delta \vdash A \rightarrow B}{\Delta, A \vdash B} \]

\[ \Delta, A_i \vdash C \]
\[ \frac{\Delta, A_1 \ast A_2 \vdash C}{\Delta, A_1 \ast A_2} \]
\[ \Delta \vdash A_1 \quad \Delta \vdash A_2 \]
\[ \frac{\Delta \vdash A_1 \ast A_2}{\Delta \vdash A_1 \ast A_2} \]

\[ \Delta, A_1 \vdash C \quad \Delta, A_2 \vdash C \]
\[ \frac{\Delta, A_1 + A_2 \vdash C}{\Delta, A_1 + A_2} \]
\[ \Delta \vdash A_i \]
\[ \frac{\Delta \vdash A_1 + A_2}{\Delta \vdash A_1 + A_2} \]

Invertible vs. non-invertible rules.
Negatives (interesting on the left): products, arrow, atoms.
Positives (interesting on the right): sums, atoms.
Invertible phase

\[
\frac{?}{X + Y \vdash X} \quad \frac{X + Y \vdash X}{X + Y \vdash X + Y}
\]

If applied too early, non-invertible rules can ruin your proof.
Invertible phase

If applied too early, non-invertible rules can ruin your proof.

**Focusing restriction 1: invertible phases**

Invertible rules must be applied as soon and as long as possible – and their order does not matter.
Invertible phase

\[ \frac{X + Y \vdash X}{\frac{X + Y \vdash X}{X + Y \vdash X + Y}} \]

If applied too early, non-invertible rules can ruin your proof.

**Focusing restriction 1: invertible phases**

Invertible rules must be applied as soon and as long as possible – and their order does not matter.

Imposing this restriction gives a single proof of \((X \rightarrow Y) \rightarrow (X \rightarrow Y)\) instead of two \((\lambda f. f \text{ and } \lambda f. \lambda x. f \ x)\).
Non-invertible phases

After all invertible rules, $\Gamma_n \vdash P_p$

Only step forward: select a formula, apply some non-invertible rules on it.
Non-invertible phases

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Only step forward: select a formula, apply some non-invertible rules on it.

Focusing restriction 2: non-invertible phase

When a principal formula is selected for non-invertible rule, they should be applied as long as possible – until its polarity changes.
Non-invertible phases

After all invertible rules, $\Gamma_n \vdash P_p$

Only step forward: select a formula, apply some non-invertible rules on it.

**Focusing restriction 2: non-invertible phase**

When a principal formula is selected for non-invertible rule, they should be applied as long as possible – until its polarity changes.

Completeness: this restriction preserves provability. **Non-trivial!**

Example of removed redundancy:

\[
\begin{array}{c}
X_2, \\
X_2 \ast X_3, \\
X_1 \ast X_2 \ast X_3,
\end{array}
\quad
\begin{array}{c}
Y_1 \vdash A \\
Y_1 \vdash A \\
Y_1 \ast Y_2 \vdash A
\end{array}
\]

\[
\begin{array}{c}
X_1 \ast X_2 \ast X_3, \quad Y_1 \ast Y_2 \vdash A
\end{array}
\]
This was focusing

Focused proofs are structured in alternating phases, invertible (boring) and non-invertible (focus).

Phases are forced to be as long as possible – to eliminate duplicate proofs.

The idea is independent from the proof system. Applies to sequent calculus or natural deduction; intuitionistic, classical, linear, you-name-it logic.

On proof terms, these restrictions correspond to $\beta\eta$-normal forms (at least for products and arrows). But the fun is in the search.
Back to unique inhabitants
You said $\beta$-short $\eta$-long normal forms?

In presence of negative connectives only (or positive only), **focused** proof search enumerates distinct normal forms.

This fails when sums (positives) are mixed with arrows (negatives).
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In presence of negative connectives only (or positive only), **focused** proof search enumerates distinct normal forms.

This fails when sums (positives) are mixed with arrows (negatives).

(Ad break)
The obvious idea...

Enumerate all derivations in a reasonable (focused) system. Remove duplicates using the equivalence algorithm. Stop if two proofs are found.
The obvious idea... does not work

Enumerate all derivations in a reasonable (focused) system. Remove duplicates using the equivalence algorithm. Stop if two proofs are found.

Infinitely many duplicates $\rightarrow$ non-termination.

\[ f : (X \rightarrow Y + Y), x : X \vdash ? : X \]
The obvious idea... does not work

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\[ f : (X \rightarrow Y + Y), x : X \vdash ? : Y \]

\[ x \]
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Enumerate all derivations in a reasonable (focused) system. Remove duplicates using the equivalence algorithm. Stop if two proofs are found.

Infinitely many duplicates $\rightarrow$ non-termination.

$$f : (X \to Y + Y), x : X \vdash \ ? : X$$

$$x$$

$$\delta(f \ x, y_1.\ x, y_2.\ x)$$
The obvious idea... does not work

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Infinitely many duplicates $\rightarrow$ non-termination.

$$f : (X \rightarrow Y + Y), \ x : X \vdash \ ? : X$$

$$x$$
$$\delta(f \ x, y_1.x, y_2.x)$$
$$\delta(f \ x, y_1.\delta(f \ x, z_1.x, z_2.x), y_2.x)$$
$$\ldots$$
The obvious idea... does not work

Enumerate all derivations in a reasonable (focused) system. Remove duplicates using the equivalence algorithm. Stop if two proofs are found.

Infinitely many duplicates $\rightarrow$ non-termination.

$f : (X \rightarrow Y + Y), x : X \vdash ? : X$

\[
\begin{align*}
x \\
\delta(f \, x, \ y_1.x, \ y_2.x) \\
\delta(f \, x, \ y_1.\delta(f \, x, \ z_1.x, \ z_2.x), \ y_2.x) \\
\ldots
\end{align*}
\]

We need a more canonical proof search process.
Terminology (a contribution?)

We consider search processes of the form “enumerating the derivations of this restricted system of inference rules”.

Various concepts of interest:
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- Provability completeness: at least one
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- Computational completeness: all programs
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Various concepts of interest:

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- Computational completeness: all programs
- Unicity completeness: at least two
Terminology (a contribution?)

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Various concepts of interest:

- Provability completeness: at least one
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- Canonicity: no duplicates — more canonical
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Focused proof search: computationally complete, not canonical. Focused proof search quotiented by equivalence: complete, canonical, non-terminating.
Terminology (a contribution?)

We consider search processes of the form “enumerating the derivations of this restricted system of inference rules”.

Various concepts of interest:

- Provability completeness: at least one
- Computational completeness: all programs
- Unicity completeness: at least two
- Canonicity: no duplicates — more canonical
- Termination — of failure

Focused proof search: computationally complete, not canonical.
Focused proof search quotiented by equivalence: complete, canonical, non-terminating.
Forward method with subsumption: provability complete but not unicity complete.
Our approach

We promised an algorithm that decides uniqueness of inhabitation. We distinguish **specification** and **implementation**.
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Specification: a novel focused logic that is **computationally complete** and **canonical**.
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We promised an algorithm that decides uniqueness of inhabitation. We distinguish **specification** and **implementation**.

Specification: a novel focused logic that is **computationally complete** and **canonical**.

Implementation: a restriction of this logic that is **unicity complete** and **terminating**.
Sum equivalence

fun f x ->
  match f x with
    ...
  fun y ->
    match f x with
      ...
      match f y with ...

Sum equivalence algorithms move case-splits up — then merge them.
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fun f x ->
  match f x with
  ...
  fun y ->
    match f x with
    ...
    match f y with ...

Sum equivalence algorithms move case-splits **up** — then merge them.

Moving up corresponds to the core idea of **maximal multi-focusing**: non-invertible phases should happen as early as possible.

http://gallium.inria.fr/~scherer/drafts/multifoc_sums.pdf
Backward search for maximal multi-focusing?

Maximality is a global property.

Building a maximal proof by goal-directed proof search seems difficult. At a focusing point $\Gamma \vdash ? : P$, we would have to guess which non-invertible phases will be used deep in $?$, and perform them now.
Backward search for maximal multi-focusing?

Maximality is a global property.

Building a maximal proof by goal-directed proof search seems difficult. At a focusing point $\Gamma \vdash ? : P$, we would have to guess which non-invertible phases will be used deep in $\Gamma$, and perform them now.

Our answer: let’s perform all the non-invertible sequences we can, even those that won’t be needed by any proof.

Saturating proof search: “Cut the positives as soon as possible”
Polarized simply-typed lambda-calculus, with non-biased atoms

\[ A, B, C, D ::= \]
\[ \mid X, Y, Z \] atoms
\[ \mid P, Q \] positive types
\[ \mid N, M \] negative types

\[ P, Q ::= A + B \] positive
\[ N, M ::= A \rightarrow B \mid A \ast B \] negative

\[ P_{at}, Q_{at} ::= P, Q \mid X, Y, Z \] positive or atomic
\[ N_{at}, M_{at} ::= N, M \mid X, Y, Z \] negative or atomic

\[ \Gamma ::= \text{varmap}(N_{at}) \] negative or atomic context
\[ \Delta ::= \text{varmap}(A) \] general context

(Note: we could have a positive product as well, it works.)
Focused natural deduction for intuitionistic logic

**INV-SUM**
\[
\begin{array}{c}
\Gamma; \Delta, x : A \vdash_{\text{inv}} t : C \\
\Gamma; \Delta, x : B \vdash_{\text{inv}} u : C
\end{array}
\]
\[
\Gamma; \Delta, x : A + B \vdash_{\text{inv}} \delta(x, x.t, x.u) : C
\]

**INV-PAIR**
\[
\begin{array}{c}
\Gamma; \Delta \vdash_{\text{inv}} t : A \\
\Gamma; \Delta \vdash_{\text{inv}} u : B
\end{array}
\]
\[
\Gamma; \Delta \vdash_{\text{inv}} (t, u) : A * B
\]

**INV-ARR**
\[
\begin{array}{c}
\Gamma; \Delta, x : A \vdash_{\text{inv}} t : B
\end{array}
\]
\[
\Gamma; \Delta \vdash_{\text{inv}} \lambda x. t : A \rightarrow B
\]

**INV-PAIR**
\[
\begin{array}{c}
\Gamma; \Delta \vdash_{\text{inv}} t : A \\
\Gamma; \Delta \vdash_{\text{inv}} u : B
\end{array}
\]
\[
\Gamma; \Delta \vdash_{\text{inv}} (t, u) : A * B
\]

**INV-END**
\[
\begin{array}{c}
\Gamma, \Gamma' \vdash_{\text{foc}} t : P_{\text{at}}
\end{array}
\]
\[
\Gamma; \Gamma' \vdash_{\text{inv}} t : P_{\text{at}}
\]
### Focused natural deduction for intuitionistic logic

**INV-SUM**

\[
\frac{\Gamma; \Delta, x : A \vdash_{\text{inv}} t : C \quad \Gamma; \Delta, x : B \vdash_{\text{inv}} u : C}{\Gamma; \Delta, x : A + B \vdash_{\text{inv}} \delta(x, x.t, x.u) : C}
\]

**INV-ARR**

\[
\frac{\Gamma; \Delta, x : A \vdash_{\text{inv}} t : B}{\Gamma; \Delta \vdash_{\text{inv}} \lambda x. t : A \rightarrow B}
\]

**INV-PAIR**

\[
\frac{\Gamma; \Delta \vdash_{\text{inv}} t : A \quad \Gamma; \Delta \vdash_{\text{inv}} u : B}{\Gamma; \Delta \vdash_{\text{inv}} (t, u) : A \times B}
\]

**INV-END**

\[
\frac{\Gamma, \Gamma' \vdash_{\text{foc}} t : P_{\text{at}}}{\Gamma; \Gamma' \vdash_{\text{inv}} t : P_{\text{at}}}
\]

**FOC-ELIM**

\[
\frac{\Gamma \vdash n \downarrow P \quad \Gamma; x : P \vdash_{\text{inv}} t : Q_{\text{at}}}{\Gamma \vdash_{\text{foc}} \text{let } x = n \text{ in } t : Q_{\text{at}}}
\]

**FOC-INTRO**

\[
\frac{\Gamma \vdash t \uparrow P}{\Gamma \vdash_{\text{foc}} t : P}
\]

**FOC-ATOM**

\[
\frac{\Gamma \vdash n \downarrow X}{\Gamma \vdash_{\text{foc}} n : X}
\]
Focused natural deduction for intuitionistic logic

INV-SUM

\[ \Gamma; \Delta, \, x : A \vdash_{\text{inv}} t : C \quad \Gamma; \Delta, \, x : B \vdash_{\text{inv}} u : C \]
\[ \Gamma; \Delta, \, x : A + B \vdash_{\text{inv}} \delta(x, \, x.t, \, x.u) : C \]

INV-PAIR

\[ \Gamma; \Delta \vdash_{\text{inv}} t : A \quad \Gamma; \Delta \vdash_{\text{inv}} u : B \]
\[ \Gamma; \Delta \vdash_{\text{inv}} (t, \, u) : A \times B \]

INV-END

\[ \Gamma, \, \Gamma' \vdash_{\text{foc}} t : P_{\text{at}} \]
\[ \Gamma; \, \Gamma' \vdash_{\text{inv}} t : P_{\text{at}} \]

FOC-ELIM

\[ \Gamma \vdash n \downarrow P \quad \Gamma; \, x : P \vdash_{\text{inv}} t : Q_{\text{at}} \]
\[ \Gamma \vdash_{\text{foc}} \text{let } x = n \text{ in } t : Q_{\text{at}} \]

FOC-INTRO

\[ \Gamma \vdash t \uparrow P \]
\[ \Gamma \vdash_{\text{foc}} t : P \]

FOC-ATOM

\[ \Gamma \vdash n \downarrow X \]
\[ \Gamma \vdash_{\text{foc}} n : X \]

INTRO-SUM

\[ \Gamma \vdash t \uparrow A_i \]
\[ \Gamma \vdash \sigma_i \, t \uparrow A_1 + A_2 \]

INTRO-END

\[ \Gamma; \emptyset \vdash_{\text{inv}} t : N_{\text{at}} \]
\[ \Gamma \vdash t \uparrow N_{\text{at}} \]
## Focused natural deduction for intuitionistic logic

### INV-SUM

\[
\begin{align*}
\Gamma; \Delta, x : A & \vdash_{\text{inv}} t : C \\
\Gamma; \Delta, x : B & \vdash_{\text{inv}} u : C \\
\Gamma; \Delta, x : A + B & \vdash_{\text{inv}} \delta(x, x.t, x.u) : C
\end{align*}
\]

### INV-ARR

\[
\begin{align*}
\Gamma; \Delta, x : A & \vdash_{\text{inv}} t : B \\
\Gamma; \Delta & \vdash_{\text{inv}} \lambda x. t : A \to B
\end{align*}
\]

### INV-PAIR

\[
\begin{align*}
\Gamma; \Delta & \vdash_{\text{inv}} t : A \\
\Gamma; \Delta & \vdash_{\text{inv}} u : B \\
\Gamma; \Delta & \vdash_{\text{inv}} (t, u) : A \times B
\end{align*}
\]

### INV-END

\[
\begin{align*}
\Gamma, \Gamma' & \vdash_{\text{foc}} t : P_{\text{at}} \\
\Gamma; \Gamma' & \vdash_{\text{inv}} t : P_{\text{at}}
\end{align*}
\]

### FOC-ELIM

\[
\begin{align*}
\Gamma & \vdash t \downarrow P \\
\Gamma; x : P & \vdash_{\text{inv}} t : Q_{\text{at}} \\
\Gamma & \vdash_{\text{foc}} \text{let} \ x = n \ \text{in} \ t : Q_{\text{at}}
\end{align*}
\]

### FOC-INTRO

\[
\begin{align*}
\Gamma & \vdash t \uparrow P \\
\Gamma & \vdash_{\text{foc}} t : P \\
\Gamma & \vdash n \downarrow X
\end{align*}
\]

### FOC-ATOM

\[
\begin{align*}
\Gamma & \vdash n \downarrow X
\end{align*}
\]

### INTRO-SUM

\[
\begin{align*}
\Gamma & \vdash t \uparrow A_i \\
\Gamma; \emptyset & \vdash_{\text{inv}} t : N_{\text{at}} \\
\Gamma & \vdash \sigma_i t \uparrow A_1 + A_2 \\
\Gamma & \vdash t \uparrow N_{\text{at}}
\end{align*}
\]

### INTRO-END

\[
\begin{align*}
\Gamma; \emptyset & \vdash_{\text{inv}} t : N_{\text{at}} \\
\Gamma & \vdash n \downarrow A \to B \\
\Gamma & \vdash u \uparrow A \\
\Gamma & \vdash n \ u \downarrow B
\end{align*}
\]

### ELIM-PAIR

\[
\begin{align*}
\Gamma & \vdash n \downarrow A_1 \times A_2 \\
\Gamma & \vdash \pi_i n \downarrow A_i \\
\Gamma & \vdash \pi_1 n \downarrow A_1 \\
\Gamma & \vdash \pi_2 n \downarrow A_2
\end{align*}
\]

### ELIM-START

\[
\begin{align*}
\Gamma; (x : N_{\text{at}}) & \in \Gamma \\
\Gamma & \vdash x \downarrow N_{\text{at}}
\end{align*}
\]

### ELIM-ARR

\[
\begin{align*}
\Gamma & \vdash n \downarrow A \to B \\
\Gamma & \vdash u \uparrow A \\
\Gamma & \vdash n \ u \downarrow B
\end{align*}
\]
Saturating focused natural deduction

\[
\text{SINV-SUM} \\
\Gamma; \Delta, x : A \vdash_{\text{sinv}} t : C \quad \Gamma; \Delta, x : B \vdash_{\text{sinv}} u : C \\
\Gamma; \Delta, x : A + B \vdash_{\text{sinv}} \delta(x, x.t, x.u) : C
\]

\[
\text{SINV-PAIR} \\
\Gamma; \Delta \vdash_{\text{sinv}} t : A \quad \Gamma; \Delta \vdash_{\text{sinv}} u : B \\
\Gamma; \Delta \vdash_{\text{sinv}} (t, u) : A \times B
\]

\[
\text{SINV-ARR} \\
\Gamma; \Delta, x : A \vdash_{\text{sinv}} t : B \\
\Gamma; \Delta \vdash_{\text{sinv}} \lambda x. t : A \rightarrow B
\]

\[
\text{SINV-END} \\
\Gamma; \Gamma' \vdash_{\text{sat}} t : P_{\text{at}} \\
\Gamma; \Gamma' \vdash_{\text{sinv}} t : P_{\text{at}}
\]
Saturating focused natural deduction

\[
\begin{align*}
\text{SINV-SUM} & : \Gamma; \Delta, x : A \vdash_{\text{sinv}} t : C \quad \Gamma; \Delta, x : B \vdash_{\text{sinv}} u : C \\
& \quad \Gamma; \Delta, x : A + B \vdash_{\text{sinv}} \delta(x, x.t, x.u) : C
\end{align*}
\]

\[
\begin{align*}
\text{SINV-PAIR} & : \Gamma; \Delta \vdash_{\text{sinv}} t : A \quad \Gamma; \Delta \vdash_{\text{sinv}} u : B \\
& \quad \Gamma; \Delta \vdash_{\text{sinv}} (t, u) : A \ast B
\end{align*}
\]

\[
\begin{align*}
\text{SINV-ARR} & : \Gamma; \Delta, x : A \vdash_{\text{sinv}} t : B \\
& \quad \Gamma; \Delta \vdash_{\text{sinv}} \lambda x. t : A \rightarrow B
\end{align*}
\]

\[
\begin{align*}
\text{SINV-END} & : \Gamma; \Gamma' \vdash_{\text{sat}} t : P_{\text{at}} \\
& \quad \Gamma; \Gamma' \vdash_{\text{sinv}} t : P_{\text{at}}
\end{align*}
\]

\[
\begin{align*}
\text{INTRO-SUM} & : \Gamma \vdash t \uparrow A_i \\
& \quad \Gamma \vdash \sigma_i t \uparrow A_1 + A_2
\end{align*}
\]

\[
\begin{align*}
\text{INTRO-END} & : \Gamma; \emptyset \vdash_{\text{sinv}} t : N_{\text{at}} \\
& \quad \Gamma \vdash t \uparrow N_{\text{at}}
\end{align*}
\]
Saturating focused natural deduction

\[
\begin{align*}
\text{SINV-SUM} & \quad \Gamma; \Delta, \ x : A \vdash_{\text{sinv}} t : C \quad \Gamma; \Delta, \ x : B \vdash_{\text{sinv}} u : C \\
& \quad \frac{}{\Gamma; \Delta, \ x : A + B \vdash_{\text{sinv}} \delta(x, x.t, x.u) : C}
\end{align*}
\]

\[
\begin{align*}
\text{SINV-PAIR} & \quad \Gamma; \Delta \vdash_{\text{sinv}} t : A \quad \Gamma; \Delta \vdash_{\text{sinv}} u : B \\
& \quad \frac{}{\Gamma; \Delta \vdash_{\text{sinv}} (t, u) : A \times B}
\end{align*}
\]

\[
\begin{align*}
\text{SINV-ARR} & \quad \Gamma; \Delta, \ x : A \vdash_{\text{sinv}} t : B \\
& \quad \frac{}{\Gamma; \Delta \vdash_{\text{sinv}} \lambda x. t : A \to B}
\end{align*}
\]

\[
\begin{align*}
\text{SINV-END} & \quad \Gamma; \Gamma' \vdash_{\text{sinv}} t : \text{P}_{\text{at}} \\
& \quad \frac{}{\Gamma; \Gamma' \vdash_{\text{sinv}} t : \text{P}_{\text{at}}}
\end{align*}
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\[
\begin{align*}
\text{INTRO-SUM} & \quad \Gamma \vdash t \uparrow A_i \\
& \quad \frac{}{\Gamma \vdash \sigma_i t \uparrow A_1 + A_2}
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\[
\begin{align*}
\text{INTRO-END} & \quad \Gamma; \emptyset \vdash_{\text{sinv}} t : \text{N}_{\text{at}} \\
& \quad \frac{}{\Gamma \vdash t \uparrow \text{N}_{\text{at}}}
\end{align*}
\]

\[
\begin{align*}
\text{ELIM-PAIR} & \quad \Gamma \vdash n \downarrow A_1 \times A_2 \\
& \quad \frac{}{\Gamma \vdash \pi_i n \downarrow A_i}
\end{align*}
\]

\[
\begin{align*}
\text{ELIM-START} & \quad (x : \text{N}_{\text{at}}) \in \Gamma \\
& \quad \frac{}{\Gamma \vdash x \downarrow \text{N}_{\text{at} 19}}
\end{align*}
\]

\[
\begin{align*}
\text{ELIM-ARR} & \quad \Gamma \vdash n \downarrow A \to B \quad \Gamma \vdash u \uparrow A \\
& \quad \frac{}{\Gamma \vdash n u \downarrow B}
\end{align*}
\]

\[
\begin{align*}
\text{ELIM-END} & \quad \Gamma \vdash \text{sat} t : \text{P}_{\text{at}} \\
& \quad \frac{}{\Gamma \vdash \text{sat} t : \text{P}_{\text{at}}}
\end{align*}
\]
Saturating focused natural deduction

\[
\text{SINV-SUM} \quad \frac{\Gamma; \Delta, x : A \vdash_{\text{sinv}} t : C \quad \Gamma; \Delta, x : B \vdash_{\text{sinv}} u : C}{\Gamma; \Delta \vdash_{\text{sinv}} \delta(x, x.t, x.u) : C}
\]

\[
\text{SINV-PAIR} \quad \frac{\Gamma; \Delta \vdash_{\text{sinv}} t : A \quad \Gamma; \Delta \vdash_{\text{sinv}} u : B}{\Gamma; \Delta \vdash_{\text{sinv}} (t, u) : A * B}
\]

\[
(\bar{n}, \bar{P}) = \{(n, P) \mid (\Gamma, \Gamma' \vdash n \downarrow P)\}
\]

\[
\Gamma, \Gamma'; \bar{x} : \bar{P} \vdash_{\text{sinv}} t : Q_{\text{at}}
\]

\[
\Gamma; \Gamma' \vdash_{\text{sat}} \text{let } \bar{x} = \bar{n} \text{ in } t : Q_{\text{at}}
\]

\[
\text{SINV-ARR} \quad \frac{\Gamma; \Delta, x : A \vdash_{\text{sinv}} t : B}{\Gamma; \Delta \vdash_{\text{sinv}} \lambda x. t : A \rightarrow B}
\]

\[
\text{SINV-END} \quad \frac{\Gamma; \Gamma' \vdash_{\text{sat}} t : P_{\text{at}}}{\Gamma; \Gamma' \vdash_{\text{sinv}} t : P_{\text{at}}}
\]

\[
\text{saturation} \quad X, (X \rightarrow Y + Y) \vdash Y
\]

\[
\text{SAT-INTRO} \quad \frac{\Gamma \vdash t \uparrow P}{\Gamma; \emptyset \vdash_{\text{sat}} t : P}
\]

\[
\text{SAT-ATOM} \quad \frac{\Gamma \vdash n \downarrow X}{\Gamma; \emptyset \vdash_{\text{sat}} n : X}
\]

\[
\text{INTRO-SUM} \quad \frac{\Gamma \vdash t \uparrow A_i}{\Gamma; \emptyset \vdash_{\text{sinv}} t : N_{\text{at}}}
\]

\[
\text{INTRO-END} \quad \frac{\Gamma \vdash n \downarrow A \rightarrow B}{\Gamma \vdash \sigma_i t \uparrow A_1 + A_2}
\]

\[
\text{ELIM-PAIR} \quad \frac{\Gamma \vdash n \downarrow A_1 * A_2}{\Gamma \vdash \pi_i n \downarrow A_i}
\]

\[
\text{ELIM-START} \quad \frac{(x : N_{\text{at}}) \in \Gamma}{\Gamma \vdash x \downarrow N_{\text{at}}}
\]

\[
\text{ELIM-ARR} \quad \frac{\Gamma \vdash n \downarrow A \rightarrow B}{\Gamma \vdash n \downarrow B}
\]

\[
\text{ELIM-ARR} \quad \frac{\Gamma \vdash u \uparrow A}{\Gamma \vdash n u \downarrow B}
\]
Saturating focused natural deduction

\[
\text{SINV-SUM} \\
\Gamma; \Delta, x : A \vdash_{\text{sinv}} t : C \\
\Gamma; \Delta, x : B \vdash_{\text{sinv}} u : C \\
\Gamma; \Delta, x : A + B \vdash_{\text{sinv}} \delta(x, x.t, x.u) : C
\]

\[
\text{SINV-PAIR} \\
\Gamma; \Delta \vdash_{\text{sinv}} t : A \\
\Gamma; \Delta \vdash_{\text{sinv}} u : B \\
\Gamma; \Delta \vdash_{\text{sinv}} (t, u) : A \times B
\]

\[
(\vec{n}, \vec{P}) = \{(n, P) \mid (\Gamma, \Gamma' \vdash n \downarrow P) \land \text{n uses } \Gamma'\} \\
\Gamma, \Gamma'; \vec{x} : \vec{P} \vdash_{\text{sinv}} t : Q_{\text{at}} \\
\Gamma; \Gamma' \vdash_{\text{sat}} \text{let } \vec{x} = \vec{n} \text{ in } t : Q_{\text{at}}
\]

\[
\text{SINV-ARR} \\
\Gamma; \Delta, x : A \vdash_{\text{sinv}} t : B \\
\Gamma; \Delta \vdash_{\text{sinv}} \lambda x. t : A \rightarrow B
\]

\[
\text{SINV-END} \\
\Gamma; \Gamma' \vdash_{\text{sat}} t : P_{\text{at}} \\
\Gamma; \Gamma' \vdash_{\text{sinv}} t : P_{\text{at}}
\]

\[
\text{canonicity} \\
X, (X \rightarrow X + X) \vdash X
\]

\[
\text{SAT-INTRO} \\
\Gamma \vdash t \uparrow P \\
\Gamma; \emptyset \vdash_{\text{sat}} t : P
\]

\[
\text{SAT-ATOM} \\
\Gamma \vdash n \downarrow X \\
\Gamma; \emptyset \vdash_{\text{sat}} n : X
\]

\[
\text{INTRO-SUM} \\
\Gamma \vdash t \uparrow A_{i} \\
\Gamma; \emptyset \vdash_{\text{sinv}} t : N_{\text{at}} \\
\Gamma \vdash \sigma_{i} t \uparrow A_{1} + A_{2}
\]

\[
\text{INTRO-END} \\
\Gamma; \emptyset \vdash_{\text{sinv}} t : N_{\text{at}} \\
\Gamma \vdash t \uparrow N_{\text{at}}
\]

\[
\text{ELIM-PAIR} \\
\Gamma \vdash n \downarrow A_{1} * A_{2} \\
\Gamma \vdash \pi_{i} n \downarrow A_{i}
\]

\[
\text{ELIM-START} \\
(x : N_{\text{at}}) \in \Gamma \\
\Gamma \vdash x \downarrow N_{\text{at}}_{19}
\]

\[
\text{ELIM-ARR} \\
\Gamma \vdash n \downarrow A \rightarrow B \\
\Gamma \vdash u \uparrow A \\
\Gamma \vdash n u \downarrow B
\]
### Saturating focused natural deduction

<table>
<thead>
<tr>
<th>Rule</th>
<th>Description</th>
</tr>
</thead>
</table>
| **SINV-SUM** | $\Gamma; \Delta, x : A \vdash_{\text{sinv}} t : C$  
\hspace{1cm} $\Gamma; \Delta, x : B \vdash_{\text{sinv}} u : C$  
\hspace{1cm} $\Gamma; \Delta, x : A + B \vdash_{\text{sinv}} \delta(x, x.t, x.u) : C$ |
| **SINV-ARR** | $\Gamma; \Delta, x : A \vdash_{\text{sinv}} t : B$  
\hspace{1cm} $\Gamma; \Delta \vdash_{\text{sinv}} \lambda x. t : A \to B$ |
| **SINV-PAIR** | $\Gamma; \Delta \vdash_{\text{sinv}} t : A$  
\hspace{1cm} $\Gamma; \Delta \vdash_{\text{sinv}} u : B$  
\hspace{1cm} $\Gamma; \Delta \vdash_{\text{sinv}} (t, u) : A \times B$ |
| **SINV-END** | $\Gamma; \Gamma' \vdash_{\text{sinv}} t : P_{\text{at}}$ |
| | $(\vec{n}, \vec{P} \subseteq \{ (n, P) \mid (\Gamma, \Gamma' \vdash n \downarrow P) \land n \text{ uses } \Gamma' \} )$  
\hspace{1cm} $\Gamma, \Gamma'; \vec{x} : \vec{P} \vdash_{\text{sinv}} t : Q_{\text{at}}$  
\hspace{1cm} $\forall x \in \vec{x}, t \text{ uses } x$  
\hspace{1cm} $\Gamma; \Gamma' \vdash_{\text{sat}} \text{let } \vec{x} = \vec{n} \text{ in } t : Q_{\text{at}}$ |
| **INTRO-SUM** | $\Gamma \vdash t \uparrow A_i$ |
| **INTRO-END** | $\Gamma; \emptyset \vdash_{\text{sinv}} t : N_{\text{at}}$ |
| **INTRO-ATOM** | $\Gamma \vdash n \downarrow X$ |
| **INTRO-PAIR** | $\Gamma \vdash n \downarrow A_1 \times A_2$ |
| **INTRO-ARR** | $\Gamma \vdash n \downarrow A \to B$ |
| **FIN-PAIR** | $\Gamma \vdash \pi_i n \downarrow A_i$ |
| **FIN-START** | $\Gamma \vdash x \downarrow N_{\text{at}}^{19}$ |
| **FIN-ARR** | $\Gamma \vdash n \downarrow A \to B$ |

#### Finite derivations

- **SAT-INTRO**  
  $\Gamma \vdash t \uparrow P$ 
  $\Gamma; \emptyset \vdash_{\text{sat}} t : P$ 

- **SAT-ATOM**  
  $\Gamma \vdash n \downarrow X$ 
  $\Gamma; \emptyset \vdash_{\text{sat}} n : X$ 

- **INTRO-SUM**  
  $\Gamma \vdash t \uparrow A_i$ 
  $\Gamma; \emptyset \vdash_{\text{sinv}} t : N_{\text{at}}$ 

- **FIN-PAIR**  
  $\Gamma \vdash n \downarrow A_1 \times A_2$ 
  $\Gamma \vdash \pi_i n \downarrow A_i$ 

- **FIN-START**  
  $\Gamma \vdash x \downarrow N_{\text{at}}^{19}$ 
  $\Gamma \vdash x \downarrow N_{\text{at}}$ 

- **FIN-ARR**  
  $\Gamma \vdash n \downarrow A \to B$ 
  $\Gamma \vdash u \uparrow A$ 

- **FIN-END**  
  $\Gamma; \emptyset \vdash_{\text{sinv}} t : N_{\text{at}}$ 

Saturating focused natural deduction

**SINV-SUM**
\[
\frac{\Gamma; \Delta, x : A \vdash_{\text{sinv}} t : C \quad \Gamma; \Delta, x : B \vdash_{\text{sinv}} u : C}{\Gamma; \Delta, x : A + B \vdash_{\text{sinv}} \delta(x, x.t, x.u) : C}
\]

**SINV-PAIR**
\[
\frac{\Gamma; \Delta \vdash_{\text{sinv}} t : A \quad \Gamma; \Delta \vdash_{\text{sinv}} u : B}{\Gamma; \Delta \vdash_{\text{sinv}} (t, u) : A \times B}
\]

**SINV-ARR**
\[
\frac{\Gamma; \Delta, x : A \vdash_{\text{sinv}} t : B}{\Gamma; \Delta \vdash_{\text{sinv}} \lambda x. t : A \to B}
\]

**SINV-END**
\[
\frac{\Gamma; \Gamma' \vdash_{\text{sat}} t : P_{\text{at}}}{\Gamma; \Gamma' \vdash_{\text{sinv}} t : P_{\text{at}}}
\]

\[
(\bar{n}, \bar{P} \subseteq \{(n, P) \mid (\Gamma, \Gamma' \vdash n \downarrow P) \land n \text{ uses } \Gamma'\})
\]

\[
\frac{\Gamma, \Gamma'; \bar{x} : \bar{P} \vdash_{\text{sinv}} t : Q_{\text{at}} \quad \forall x \in \bar{x}, t \text{ uses } x}{\Gamma; \Gamma' \vdash_{\text{sat}} \text{let } \bar{x} = \bar{n} \text{ in } t : Q_{\text{at}}}
\]

**SAT-INTRO**
\[
\frac{\Gamma \vdash t \uparrow P}{\Gamma; \emptyset \vdash_{\text{sat}} t : P}
\]

**SAT-ATOM**
\[
\frac{\Gamma \vdash n \downarrow X}{\Gamma; \emptyset \vdash_{\text{sat}} n : X}
\]

**INTRO-SUM**
\[
\frac{\Gamma \vdash t \uparrow A_i}{\Gamma; \emptyset \vdash_{\text{sinv}} t : N_{\text{at}}}
\]

**INTRO-END**
\[
\frac{\Gamma \vdash n \downarrow A \to B \quad \Gamma \vdash u \uparrow A}{\Gamma \vdash n.u \downarrow B}
\]

**INTRO-END**
\[
\frac{\Gamma \vdash t \uparrow A_1 + A_2}{\Gamma \vdash \sigma_i t \uparrow A_1 + A_2}
\]

**ELIM-PAIR**
\[
\frac{\Gamma \vdash n \downarrow A_1 \times A_2}{\Gamma \vdash \pi_i n \downarrow A_i}
\]

**ELIM-START**
\[
\frac{(x : N_{\text{at}}) \in \Gamma}{\Gamma \vdash x \downarrow N_{\text{at}}^{19}}
\]

**ELIM-ARR**
\[
\frac{\Gamma \vdash n \downarrow A \to B \quad \Gamma \vdash u \uparrow A}{\Gamma \vdash n.u \downarrow B}
\]
What we have so far:

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What we lack:

- termination
Usual termination arguments

Subformula property: finite number of judgments $\Gamma \vdash A$. 
Usual termination arguments

Subformula property: finite number of judgments $\Gamma \vdash A$.

No need for recurrent judgments.

\[
\begin{array}{c}
? \\
\hline \Gamma \vdash A \\
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\hline \Gamma \vdash A \\
\cdots 
\end{array}
\]
Usual termination arguments

Subformula property: finite number of judgments $\Gamma \vdash A$. Not true for typing contexts (multisets)

No need for recurrent judgments.

\[
\begin{array}{c}
? \\
\Gamma \vdash A \\
\vdash\vdash \vdash \\
\Gamma \vdash A \\
\vdash \\
\end{array}
\]
Usual termination arguments

Subformula property: finite number of judgments $\Gamma \vdash A$. Not true for typing contexts (multisets)

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\[
\begin{array}{c}
? \\
\hline
\Gamma \vdash A \\
\ldots \\
\ldots \\
\ldots \\
\hline
\Gamma \vdash A \\
\ldots \\
\end{array}
\]

Breaks computational completeness
Termination 1: bounded multisets

There exists a $n \in \mathbb{N}$ such that, by keeping at most $n$ variables of each type/formula in $\Gamma$, then we can find at least 2 distinct proofs of $\Gamma \vdash A$ if they exist.
Termination 1: bounded multisets

There exists a \( n \in \mathbb{N} \) such that, by keeping at most \( n \) variables of each type/formula in \( \Gamma \), then we can find at least 2 distinct proofs of \( \Gamma \vdash A \) if they exist.

In fact \( n := 2 \) suffices – for any bound \( n \), you find at least \( n \) proofs.
Termination 2: recurring at most twice

\[ \Gamma \vdash A \]

\[ \Gamma \vdash A \]

\[ \Gamma \vdash A \]

Computational completeness is lost, but unicity completeness regained.
Termination 2: recurring at most twice

\[
\begin{align*}
\text{?} \\
\Gamma \vdash A \\
\ldots \\
\ldots \ldots \ldots \\
\Gamma \vdash A \\
\ldots
\end{align*}
\]

Computational completeness is lost, but Unicity completeness regained.
Our algorithm searches for all saturated proofs under these two search-space restriction.

Optimization 1: redundancy (elim and intro).
Optimization 2: monotonicity.
(The logic again)

\[
\begin{array}{l}
\text{SINV-SUM} \\
\Gamma; \Delta, x : A \vdash_{\text{sinv}} t : C & \quad \Gamma; \Delta, x : B \vdash_{\text{sinv}} u : C \\
\Gamma; \Delta, x : A + B \vdash_{\text{sinv}} \delta(x, x.t, x.u) : C
\end{array}
\]

\[
\begin{array}{l}
\text{SINV-PAIR} \\
\Gamma; \Delta \vdash_{\text{sinv}} t : A & \quad \Gamma; \Delta \vdash_{\text{sinv}} u : B \\
\Gamma; \Delta \vdash_{\text{sinv}} (t, u) : A \ast B
\end{array}
\]

\[
\begin{array}{l}
\Gamma, \bar{n}, \bar{P} \subseteq \{(n, P) \mid (\Gamma, \Gamma' \vdash n \downarrow P) \land n \text{ uses } \Gamma'\} & \quad \text{SAT-INTRO} \\
\Gamma, \Gamma', \bar{x}, \bar{P} \vdash_{\text{sinv}} t : Q_{\text{at}} & \quad \forall x \in \bar{x}, t \text{ uses } x \\
\Gamma; \Gamma' \vdash_{\text{sat}} \text{let } \bar{x} = \bar{n} \text{ in } t : Q_{\text{at}}
\end{array}
\]

\[
\begin{array}{l}
\text{SAT-ATOM} \\
\Gamma \vdash n \downarrow X \\
\Gamma; \emptyset \vdash_{\text{sat}} n : X
\end{array}
\]

\[
\begin{array}{l}
\text{INTRO-SUM} \\
\Gamma \vdash t \uparrow A_i \\
\Gamma \vdash \sigma_i t \uparrow A_1 + A_2
\end{array}
\]

\[
\begin{array}{l}
\text{INTRO-END} \\
\Gamma; \emptyset \vdash_{\text{sinv}} t : N_{\text{at}} \\
\Gamma \vdash t \uparrow N_{\text{at}}
\end{array}
\]

\[
\begin{array}{l}
\text{ELIM-PAIR} \\
\Gamma \vdash n \downarrow A_1 \ast A_2 \\
\Gamma \vdash \pi_i n \downarrow A_i
\end{array}
\]

\[
\begin{array}{l}
\text{ELIM-START} \\
(x : N_{\text{at}}) \in \Gamma \\
\Gamma \vdash x \downarrow N_{\text{at}}
\end{array}
\]

\[
\begin{array}{l}
\text{ELIM-ARR} \\
\Gamma \vdash n \downarrow A \rightarrow B & \quad \Gamma \vdash u \uparrow A \\
\Gamma \vdash n u \downarrow B
\end{array}
\]
Conclusion

Future work: extend to polymorphism and dependent types.
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Thanks. Any question?

Paper draft:  
gallium.inria.fr/~scherer/drafts/unique_stlc_sums.pdf