Normalization by realizability also evaluates

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Acquire a better understanding of "semantic soundness proofs" for type systems: realizability and logical relations.

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Answer: an evaluation program.

The result appears to be not-well-communicated folklore.
We will (briefly) discuss related works.

Setting

We will look at a soundness proof:

- of weak normalization
- for the simply-typed lambda-calculus
- using classical realizability

$$\vdash t : A \quad \Longrightarrow \quad t \in |A|$$

If t is well-typed at A, then it belongs to the set $|A|$ of "good terms".

Classical realizability in one slide

A soundness technique for *abstract machines* formed of a pair $\langle\, t \mid e \,\rangle$ (in $\mathbb{M}$) of a *term* t (in $\mathbb{T}$) and a *co-term* (context) e (in $\mathbb{E}$).

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For the right definitions, we prove an *adequacy lemma* saying that:

- well-typed terms $t : A$ belong to a set of *truth witnesses* $|A|$
- well-typed co-terms $e : A$ belong to a set of *falsity witnesses* $\|A\|$
- well-typed machines (combining those) belong to a *pole* $\perp\!\!\!\perp$.

Those sets capture *good* (sound) terms/coterms/machines.
Here, we define $\perp\!\!\!\perp$ as the set of machines that reduce to a valid machine in normal form.

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We will define $|A|$ and $\|A\|$ such that $t \in |A|$ and $e \in \|A\|$ imply $\langle t \mid e \rangle \in \perp\!\!\!\perp$.

Orthogonality is central to this:

$$\mathcal{T}^{\perp} \triangleq \{e \mid \forall t \in \mathcal{T},\ \langle t \mid e \rangle \in \perp\!\!\!\perp\} \qquad \mathcal{E}^{\perp} \triangleq \{t \mid \forall e \in \mathcal{E},\ \langle t \mid e \rangle \in \perp\!\!\!\perp\}$$

Concretely

Our language:

$$t \triangleq x \mid \lambda x.\, t \mid t\ u$$
$$e \triangleq \star \mid u \cdot e$$

(+ some reduction relation $\leadsto$)

normal machines: $\mathbb{M}_N \quad \triangleq \quad \langle\, x \mid e \,\rangle \ \mid\ \langle\, t \mid \star \,\rangle$

Recall that $\perp\!\!\!\perp$ is the set of machines that reduce to a normal machine.

t is weakly-normalising as a lambda-term exactly if $\langle\, t \mid \star \,\rangle$ is in $\perp\!\!\!\perp$.

Witnesses

The function type $A \to B$ is a *negative* type.
Its is determined by its *falsity witnesses* that are *values*: $\|A \to B\|_V$.
The rest follows by orthoginality. For example:

$$\begin{aligned} \|A \to B\|_V &\triangleq |A| \cdot \|B\|_V \\ |A \to B| &\triangleq \|A \to B\|_V^{\perp} \\ \|A \to B\| &\triangleq |A \to B|^{\perp} \end{aligned}$$

For a positive type we would have, for example:

$$|A \times B|_V \triangleq |A|_V * |B|_V$$

In general, for negatives N and positives P we have:

$$\begin{aligned} \|P\| &\triangleq |P|_V^{\perp} & \qquad |P| &\triangleq |P|_V^{\perp\perp} \\ \|N\| &\triangleq \|N\|_V^{\perp\perp} & \qquad |N| &\triangleq \|N\|_V^{\perp} \end{aligned}$$

General approach

We turn the proposition $\langle\, t \mid e \,\rangle \in \perp\!\!\!\perp$ into a datatype of *concrete evidence*:

$$(\_ \in \perp\!\!\!\perp) : \mathbb{M} \to \mathsf{Type}$$

$$m \in \perp\!\!\!\perp \triangleq (\Sigma([m_1, \ldots, m_n] : \mathsf{List}(\mathbb{M})).\ m \leadsto m_1 \leadsto \ldots \leadsto m_n \in \mathbb{M}_N)$$

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Truth and falsity value witnesses have specific shapes:

$$\|A \to B\|_V \triangleq |A| \times \|B\|_V$$

$$\pi_0 \in \|A \to B\|_V \triangleq \Sigma(u, \pi).\, \pi_0 \equiv u \cdot \pi \wedge u \in |A| \wedge \pi \in \|B\|_V$$

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The notion of orthogonality is also made computational:

$$\mathcal{T}^{\perp} \triangleq \{e \mid \forall t \in \mathcal{T},\ \langle\, t \mid e \,\rangle \in \perp\!\!\!\perp\}$$

$$t \in |||A^{\perp} \triangleq \Pi(e : \mathbb{E}).\, e \in |||A \to \langle\, t \mid e \,\rangle \in \perp\!\!\!\perp$$

Conclusion

We are done: the way we defined truth and value witnesses (the shape of values) *completely determines* the evaluation strategy and its implementation.

We found it rather fun – I'll try to show you a bit of it.

Simplification

$m \in \perp\!\!\!\perp$ is dependent on the machine m, $t \in |A|$ on t, etc.
As a first step, we can remove this dependency by definiting, for each predicate $\_ \in T$, a non-dependent type $\mathcal{J}(T)$.

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Adequacy, computationally

$$\texttt{rea} : \forall\ \{\Gamma\}\ t\ \{A\}\ \{\rho\}.\ \{\Gamma \vdash t : A\}\ \rightarrow \rho \in |\Gamma| \rightarrow t[\rho] \in |A|$$

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(now let's un-simplify things)

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(Slightly) more in the paper

We can change the definition of truth and value witnesses. For example:

(old) $\|A \to B\|_V \triangleq |A| * \|B\|_V$ (new) $\|A \to B\|_V \triangleq |A|_V * \|B\|_V$

$$|A * B|_V \triangleq |A| * |B| \qquad\qquad |A * B|_V \triangleq |A|_V * |B|_V$$

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When we have both positive and negative types, some definitions are by case-distinction on the polarity.
Hints of a *polarized* evaluation order.

Strongly related work

Hugo Herbelin (informally) explains that realizability and normalization-by-evaluation (NbE) are two sides of the same coin.

$$(rea) \quad \vdash t : A \rightarrow t \in |A|$$

$$(NbE) \quad (\vdash t : A \rightarrow \Vdash A) \;\wedge\; (\Vdash A \rightarrow \{v\ \mathrm{NF} \mid \vdash v : A\})$$

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In a hidden part of “Continuation-passing style models complete for intuitionistic logic” (2013), Danko Ilik remarks that the completeness proof of his Kripke-model construction (in CPS style) extracts to a NbE algorithm. He points out that a different CPS translation gives call-by-value instead of call-by-name.

Full CBN version

$$\begin{array}{lcl} \langle \_ \mid \_ \rangle_A & : & \mathcal{J}(|A|) \to \mathcal{J}(\|A\|) \to \mathcal{J}(\perp\!\!\!\perp) \\ \langle \bar{t} \mid \bar{e} \rangle_P & \triangleq & \bar{t}\ \bar{e} \\ \langle \bar{t} \mid \bar{e} \rangle_N & \triangleq & \bar{e}\ \bar{t} \end{array}$$

$$\begin{array}{llcl} \texttt{rea} & x^A & \bar{\rho} & \triangleq & \bar{\rho}(x) \\ \texttt{rea} & (\lambda x^A.\, t^B) & \bar{\rho} & \triangleq & \lambda(\bar{u}^{|A|}, \bar{e}^{\|B\|}).\, \langle\, \texttt{rea}\ t\ \bar{\rho}[x \mapsto \bar{u}] \mid \bar{e}\,\rangle_B \\ \texttt{rea} & (t^{A \to B}\ u^A) & \bar{\rho} & \triangleq & \lambda\bar{\pi}^{\|B\|_V}.\, \texttt{rea}\ t\ \bar{\rho}\ (\texttt{rea}\ u\ \bar{\rho}, (\bar{\pi})_V) \\ \texttt{rea} & (t^A, u^B) & \bar{\rho} & \triangleq & (\texttt{rea}\ t\ \bar{\rho}, \texttt{rea}\ u\ \bar{\rho})^{\perp\perp} \end{array}$$

$$\texttt{rea}\ (\texttt{let}\ (x, y) = t^{A*B}\ \texttt{in}\ u^C)\ \bar{\rho}\ \triangleq$$

$$\lambda\bar{\pi}^{\|C\|_V}.\, \langle\, \texttt{rea}\ t\ \bar{\rho} \mid \lambda(\bar{x}, \bar{y}).\ \texttt{rea}\ u\ \bar{\rho}[x \mapsto \bar{x}, y \mapsto \bar{y}]\ \bar{\pi}\,\rangle_{A*B}$$

Auxiliary definitions

$$\begin{array}{lcl} \_^{\perp\perp} & : & \mathcal{J}(|P|_V) \to \mathcal{J}(|P|) \\ (\bar{v}^{|P|_V})^{\perp\perp} & \triangleq & \lambda \bar{e}^{\|P\|}.\, \bar{e}\ \bar{v} \end{array}$$

$$\begin{array}{lcl} \_^{\perp\perp} & : & \mathcal{J}(\|N\|_V) \to \mathcal{J}(\|N\|_V) \\ (\bar{\pi}^{\|N\|_V})^{\perp\perp} & \triangleq & \lambda \bar{t}^{|N|}.\, \bar{t}\ \bar{\pi} \end{array}$$

$$\begin{array}{lcl} (\_)_V & : & \mathcal{J}(|A|_V) \to \mathcal{J}(|A|) \\ (\bar{v}^{|P|_V})_V & \triangleq & \bar{v}^{\perp\perp} \\ (\bar{t}^{|N|_V})_V & \triangleq & \bar{t} \end{array}$$

$$\begin{array}{lcl} (\_)_V & : & \mathcal{J}(\|A\|_V) \to \mathcal{J}(\|A\|) \\ (\bar{e}^{\|P\|_V})_V & \triangleq & \bar{e} \\ (\bar{\pi}^{\|N\|_V})_V & \triangleq & \bar{\pi}^{\perp\perp} \end{array}$$

CBV arrow

$$
\begin{array}{llcl}
\texttt{rea} & x^A & \bar\rho & \triangleq & (\bar\rho(x))_V \\
\texttt{rea} & (\lambda x^A.\, t^B) & \bar\rho & \triangleq & \lambda(\bar v^{|A|_V}, \bar e^{\|B\|}).\, \langle\, \texttt{rea}\ t\ \bar\rho[x \mapsto \bar v] \mid \bar e\,\rangle_B \\
\texttt{rea} & (t^{A\to B}\ u^A) & \bar\rho & \triangleq & \lambda \bar\pi^{\|B\|_V}.\, \langle\, \texttt{rea}\ u\ \bar\rho \mid \lambda \bar v_u^{|A|_V}.\,\texttt{rea}\ t\ \bar\rho\ (\bar v_u, (\bar\pi)_V)\,\rangle_A
\end{array}
$$