Functional programming with λ−tree syntax

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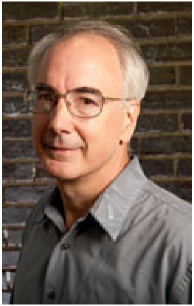


Introduction

MLTS is an ongoing language design experiment. WIP!
Extend ML with binder handling constructs from λProlog and Abella.

Theory: in logic programming, computation from *proof search*.
Binders: a new *quantifier* in the logic: ∇x, "for a fresh x".

Implementation: online, compiles to λProlog.
`https://voodoos.github.io/mlts/`

Look and feel: a funny mix of FreshML and HOAS.
Mobility and λ-Tree Syntax.

MLTS: datatypes with binders

MLTS extends ML with binders.

Normal ML datatypes are *closed*.

Example of *open* datatype:

```
type lam =
| App of lam * lam
| Abs of lam => lam
;;
```

(notice: no constructor for variables)

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Inhabitants:

$\lambda x.\ x$	`Abs(X \ X)`
$\lambda x.\ (x\ x)$	`Abs(X \ App(X, X))`
$(\lambda x.\ x)\ (\lambda x.\ x)$	`App(Abs(X \ X), Abs(X \ X))`

MLTS crash course

`subst : lam -> lam -> lam`

`subst t x u` is $t[x\backslash u]$.

```
let rec subst t x u = match (x, t) with
```

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`nab X in (X, X)` will only match if `x = t = X` is a nominal.

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`subst : lam -> lam -> lam`

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`nab X Y in (X, Y)` will only match two distinct nominals.

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let rec subst t x u = match (x, t) with
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`r @ Y : lam`

`Abs(Y\ r @ Y) : lam`

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`subst : lam -> lam -> lam`

`subst t x u` is $t[x\backslash u]$.

In `Abs(Y \ subst (r @ Y) x u)`, no variable is ever free.
Binders move.

```
let rec subst t x u = match (x, t) with
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Binder type

`(a => b)`: "open" values of type `b` under a binder of type `a`.

introduction `X \ t`, elimination `t @ X`.

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(and in patterns)

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`new X in`: a scope in which a new nominal `X` is available.

Effect: escape checking / occurs check.
(Safer when returning a closed type.)

Pure substitution

$$\frac{\Gamma, x \vdash t \qquad \Gamma \vdash u}{\Gamma \vdash t[u/x]}$$

```
let rec subst (t : lam => lam) (u : lam) : lam =
  match t with
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  | X \ Abs (Y \ r @ X Y) ->
      Abs (Y \ subst (X \ r @ X Y) u)
```

Beta reduction

```
let rec beta t =
  match t with
  | nab X in X -> X
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let rec beta t =
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Pattern matching

Unification modulo α, β_0 and η.
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Implied restrictions:

- Applications lists are distinct nominals.
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```

This is called higher-order pattern unification.
Decidable, most general unifiers.

Interpreter in λProlog: just ML

ML admits type-erasure:

can define an operational semantics on untyped terms.

$\Longrightarrow$ untyped interpreter in λ-prolog, all ML types map to tm.

```
kind tm         type.
type app        tm -> tm -> tm.
type lam        (tm -> tm) -> tm.
type let        tm -> (tm -> tm) -> tm.
type match      tm -> clauses -> tm.

type K          tm -> ... -> tm -> tm.

type cp         tm -> tm -> prop.
type eval       tm -> tm -> prop.

eval (let Def Body) VB :-
  eval Def VD,
  eval (Body VD) VB.
```

Interpreter in λProlog: MLTS

To extend to MLTS,

$$\texttt{transl(a => b)} = \texttt{tm -> transl(b)} \qquad \texttt{transl(\_)} = \texttt{tm}$$

Interpreter in λProlog: MLTS

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transl(`a => b`) = `tm -> ` transl(`b`)   transl(`_`) = `tm`

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To extend to MLTS,

$$\text{transl}(\texttt{a => b}) = \texttt{tm -> } \text{transl}(\texttt{b}) \qquad \text{transl}(\_) = \texttt{tm}$$

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type new_τ    (tm -> τ) -> τ.
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```
type new_τ    (tm -> τ) -> τ.
```

```
eval (new_τ T) V :- pi x \ eval (T x) V.
```

Demo time?

Implementation by Ulysse Gérard.

Technology: Menhir + his code + Elpi + `js_of_ocaml` + Nice web stuff.

Conclusion & Future work

- This treatment of bindings has a clean semantic inspired by Abella.
- The interpreter was quite simple to write : ≈140 lines of code

Future work:

- More examples in the meta-programming area (a compiler ?)
- Provide an operational semantics (small-step?) without primitive binding constructs.
- Statics checks such as pattern matching exhaustivity, use of distinct pattern variables in pattern application, nominals escaping their scope, etc.
- Design a "real" implementation. A compiler ? An extension to OCaml ? An abstract machine ?

https://trymlts.github.io

Thank you

Concrete syntax typing rules (1/2)

$$\frac{}{\Gamma, x : C \vdash \texttt{x} : \texttt{C}} \qquad \frac{\Gamma \vdash \texttt{M} : \texttt{A -> B} \qquad \Gamma \vdash \texttt{N} : \texttt{A}}{\Gamma \vdash \texttt{(M N)} : \texttt{B}}$$

$$\frac{\Gamma, \texttt{x} : \texttt{A} \vdash \texttt{M} : \texttt{B}}{\Gamma \vdash \texttt{(fun x -> M)} : \texttt{A -> B}}$$

$$\frac{\Gamma, \texttt{X} : \texttt{A} \vdash \texttt{M} : \texttt{B} \qquad \text{open } \texttt{A}}{\Gamma \vdash \texttt{(new X in M)} : \texttt{B}} \qquad \frac{\Gamma, \texttt{X} : \texttt{A} \vdash \texttt{M} : \texttt{B} \qquad \text{open } \texttt{A}}{\Gamma \vdash \texttt{(X \textbackslash{} M)} : \texttt{A => B}}$$

$$\frac{\Gamma \vdash \texttt{r} : \texttt{A1 => ... => An => A} \quad \Gamma \vdash \texttt{t1} : \texttt{A1} \quad \ldots \quad \Gamma \vdash \texttt{tn} : \texttt{An}}{\Gamma \vdash \texttt{(r @ t1 ... tn)} : \texttt{A}}$$

Concrete syntax typing rules (2/2)

$$\frac{\Gamma \vdash \texttt{term} : \texttt{B} \qquad \Gamma \vdash \texttt{B} : \texttt{R1} : \texttt{A} \qquad \ldots \qquad \Gamma \vdash \texttt{B} : \texttt{Rn} : \texttt{A}}{\Gamma \vdash \texttt{match term with R1 | ... | Rn} : \texttt{A}}$$

$$\frac{\Gamma, \texttt{X} : \texttt{C} \vdash \texttt{A} : \texttt{R} : \texttt{B} \qquad \text{open } \texttt{C}}{\Gamma \vdash \texttt{A} : \texttt{nab X in R} : \texttt{B}} \qquad \frac{\Gamma \vdash \texttt{L} : \texttt{A} \vdash \Delta \qquad \Gamma, \Delta \vdash \texttt{R} : \texttt{B}}{\Gamma \vdash \texttt{A} : \texttt{L -> R} : \texttt{B}}$$

$$\frac{\Gamma \vdash \texttt{t1} : \texttt{A1} \vdash \Delta_1 \qquad \ldots \qquad \Gamma \vdash \texttt{tn} : \texttt{An} \vdash \Delta_n}{\Gamma \vdash \texttt{C(t1,...,tn)} : \texttt{A} \vdash \Delta_1, \ldots, \Delta_n} \; C \text{ of type } \texttt{A1*...*An -> A}$$

$$\frac{\Gamma \vdash \texttt{X1} : \texttt{A1} \;\ldots\; \Gamma \vdash \texttt{Xn} : \texttt{An} \qquad \text{open } \texttt{A1} \ldots \text{open } \texttt{An}}{\Gamma \vdash \texttt{(r @ X1 ... Xn)} : \texttt{A} \vdash \texttt{r} : \texttt{A1 => ... => An => A}}$$

$$\frac{}{\Gamma \vdash \texttt{x} : \texttt{A} \vdash \{\texttt{x} : \texttt{A}\}} \qquad \frac{\Gamma \vdash \texttt{p} : \texttt{A} \vdash \Delta_1 \qquad \Gamma \vdash \texttt{q} : \texttt{B} \vdash \Delta_2}{\Gamma \vdash \texttt{(p,q)} : \texttt{A * B} \vdash \Delta_1, \Delta_2}$$

Natural semantics for the abstract syntax ($\mathcal{G}$-logic [Gacek, 2009, Gacek et al., 2011]) (1/2)

$$\frac{\vdash \mathit{val}\ V}{\vdash V \Downarrow V} \qquad \frac{\vdash M \Downarrow F \quad \vdash N \Downarrow U \quad \vdash \mathit{apply}\ F\ U\ V}{\vdash M@N \Downarrow V}$$

$$\frac{\vdash (R\ U) \Downarrow V}{\vdash \mathit{apply}\ (\mathrm{lam}\ R)\ U\ V} \qquad \frac{\vdash (R\ (\mathit{fixpt}\ R)) \Downarrow V}{\vdash (\mathit{fixpt}\ R) \Downarrow V}$$

$$\frac{\vdash C \Downarrow \mathit{tt} \quad \vdash L \Downarrow V}{\vdash \mathit{cond}\ C\ L\ M \Downarrow V} \qquad \frac{\vdash C \Downarrow \mathit{ff} \quad \vdash M \Downarrow V}{\vdash \mathit{cond}\ C\ L\ M \Downarrow V}$$

Natural semantics for the abstract syntax (2/2)

$$\frac{\vdash \nabla x.(E\ x) \Downarrow (V\ x)}{\vdash x\backslash\ E\ x \Downarrow x\backslash\ V\ x} \qquad \frac{\vdash \nabla x.(E\ x) \Downarrow V}{\vdash \mathit{new}\ E \Downarrow V}$$

$$\frac{\vdash \text{pattern}\ \ T\ \ \mathit{Rule}\ \ U \qquad \vdash U \Downarrow V}{\vdash (\text{match}\ \ T\ \ (\mathit{Rule} :: \mathit{Rules})) \Downarrow V} \qquad \frac{\vdash (\text{match}\ \ T\ \ \mathit{Rules}) \Downarrow V}{\vdash (\text{match}\ \ T\ \ (\mathit{Rule} :: \mathit{Rules})) \Downarrow V}$$

$$\frac{\vdash \exists x.\text{pattern}\ \ T\ \ (P\ \ x)\ \ U}{\vdash \text{pattern}\ \ T\ \ (\mathit{all}\ \ (x\backslash\ \ P\ \ x))\ \ U} \qquad \frac{\vdash (\lambda z_1 \ldots \lambda z_m.(t \Longrightarrow s)) \trianglerighteq (T \Longrightarrow U)}{\vdash \text{pattern}\ \ T\ \ (\mathit{nab}\ z_1 \ldots \mathit{nab}\ z_m.(t \Longrightarrow s))\ \ U}$$

$$\frac{\dfrac{\vdash \lambda X.(X \Longrightarrow s) \trianglerighteq (Y \Longrightarrow U)}{\vdash \text{pattern}\ \ Y\ \ (\text{nab}\ X\ \text{in}\ (X \Longrightarrow s))\ \ U} \qquad \vdash U \Downarrow V}{\vdash \text{match}\ \ Y\ \ \text{with}\ \ \ (\text{nab}\ X\ \text{in}\ (X \Longrightarrow s)) \Downarrow V}$$

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