Polarised Intermediate Representation of λ-Calculus with Sums

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The "no simpler" problem

As simple as possible, but no simpler.

When picking a formal system to study, am I faithfully modeling the problem at hand, or reducing its complexity in essential ways?

Experience $\implies$ important features that reveal pain points

More features $\implies$ clutter risk (n^2); need a very regular presentation

This talk:

- For program equivalence, sums (positives) are essential.
- Polarized $\mu\bar{\mu}$ is a good, regular syntax for programs.

$\beta\eta$ equivalence

$$(\lambda x.t)\, u \rightarrow_\beta t[u/x] \qquad (t : A \rightarrow B) \rightarrow_\eta \lambda x.t\, x$$

$$\pi_i\,(t_1, t_2) \rightarrow_\beta t_i \qquad (t : A \times B) \rightarrow_\eta (\pi_1\, t, \pi_2\, t)$$

$\beta\eta$ equivalence

$$(\lambda x.t)\,u \to_\beta t[u/x] \qquad (t : A \to B) \to_\eta \lambda x.t\,x$$

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$$\delta(\sigma_i\,t,\ x_1.u_1,\ x_2.u_2) \to_\beta u_i[t/x_i]$$

$$(t : A + B) \to_\eta \delta(t,\ x_1.\sigma_1\,x_1,\ x_2.\sigma_2\,x_2)$$

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$$(t, u) \overset{?}{\simeq} \delta(t,\ x_1.(\sigma_1\, x_1, u),\ x_2.(\sigma_2\, x_2, u))$$

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$$\delta(\sigma_i\, t,\, x_1.u_1,\, x_2.u_2) \to_\beta u_i[t/x_i]$$

$$\forall K[\square], \quad K[t] \to_\eta \delta(t,\, x_1.K[\sigma_1\, x_1],\, x_2.K[\sigma_2\, x_2])$$

Sums seem to be trouble-makers.

Natural deduction, λ-calculus are irregular

$$\frac{\Gamma \vdash A \to B \qquad \Gamma \vdash A}{\Gamma \vdash B} \qquad\qquad \frac{\Gamma \vdash A_1 \times A_2}{\Gamma \vdash A_i}$$

$$\frac{\Gamma \vdash A + B \qquad \Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma \vdash C}$$

Natural deduction, λ-calculus are irregular

$$\begin{array}{cc} & \delta(t, x_1.(\lambda y.u_1), x_2.(\lambda y.u_2)) \\ \stackrel{?}{\simeq} & (\lambda y.\delta(t, x_1.u_1, x_2.u_2)) \end{array}$$

Natural deduction, λ-calculus are irregular

$$\begin{aligned} & \delta(t,\, x_1.(\lambda y.u_1),\, x_2.(\lambda y.u_2)) \\ \overset{?}{\simeq} \quad & (\lambda y.\delta(t,\, x_1.u_1,\, x_2.u_2)) \end{aligned}$$

$$\begin{aligned} & ((\lambda y.u_1), (\lambda y.u_2)) \\ \overset{?}{\simeq} \quad & \lambda y.(u_1, u_2) \end{aligned}$$

$$\begin{aligned} & \pi_1\,(\lambda y.u) \\ \overset{?}{\simeq} \quad & \lambda y.\pi_1\, u \end{aligned}$$

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$$\begin{array}{rc} & ((\lambda y.u_1), (\lambda y.u_2)) \\ \overset{?}{\simeq} & \lambda y.(u_1, u_2) \end{array} \qquad \begin{array}{rc} & \pi_1\,(\lambda y.u) \\ \overset{?}{\simeq} & \lambda y.\pi_1\, u \end{array}$$

Which exchanges are "allowed"? List all possibilities?

Goal / Contribution

Goal: a regular syntax of terms, in which equivalence can be elegantly expressed.

My take on our work: polarized $\mu\bar{\mu}$, as studied in Guillaume Munch-Maccagnoni's PhD thesis, provides such a syntax.

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Abstract machine

$\mu\bar{\mu}$ programs are **commands** c, built as pairs $\langle t \,\|\, e \rangle$ of a **term** t and **context** e.

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$$\begin{array}{lcl} \langle t\ u \parallel e \rangle & \to_{\mathsf{R}} & \langle t \parallel u \cdot e \rangle \\ \langle \lambda x.t \parallel u \cdot e \rangle & \to_{\mathsf{R}} & \langle t[u/x] \parallel e \rangle \end{array}$$

$(u \cdot e)$ is the "important" part that $\lambda x.t$ destructs.

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Idea 1: $(t\ u)$ is just syntactic sugar for a term $(e \mapsto \langle t \,\|\, u \cdot e\rangle)$.
Let us write this $\mu\alpha.\,\langle t \,\|\, u \cdot \alpha\rangle$.

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$$\langle \mu\alpha.\, c \parallel e\rangle \to_{\mathsf{R}} c[e/\alpha]$$

Machines with sub-machines: abstract machine **calculus**

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Idea 2: destructor syntax for $\lambda x.t$

$$\langle \mu(x \cdot \alpha).\, c \parallel u \cdot e\rangle \to_{\mathsf{R}} c[e/\alpha, u/x]$$

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$$\langle \mu(x \cdot \alpha).\, c \parallel u \cdot e\rangle \to_{\mathsf{R}} c[e/\alpha, u/x] \qquad\qquad (\lambda x.t) \stackrel{\mathtt{def}}{=} \mu(x \cdot \alpha).\, \langle t \parallel \alpha\rangle$$

The regularity of duality

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$$\begin{aligned}
\langle \mu\alpha.\, c \,\|\, e \rangle &\to_{\mathsf{R}} \quad c[e/\alpha] \\
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\langle \mu(x \cdot \alpha).\, c \parallel t \cdot e \rangle &\to_{\mathsf{R}} \quad c[e/\alpha, t/x] \\
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(Under the hood: confluence, polarization)

Taking control (away)

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Intuitionistic restriction: one single co-variable $\star$, binding occurences shadow each other.

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$$\mu \star . \left\langle t \,\middle\|\, (\mu(x \cdot \star). \langle x \,\|\, \star \rangle) \cdot \star \right\rangle$$

Reductions, expansions

Analog of η-expansion rules such as $t \simeq_\eta \lambda x.\,(\;t\quad x\;)$

$$t \quad \rightarrow_{\mathsf{E}}$$

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$$t \quad \to_{\mathsf{E}} \quad \mu(x \cdot \star).$$

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Reductions, expansions

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$$
\begin{array}{lll}
t & \to_{\mathsf{E}} & \mu(x \cdot \star).\,\langle t \,\|\, x \cdot \star\rangle \\
e & \to_{\mathsf{E}} & \bar{\mu}(x, y).\,\langle (x, y) \,\|\, e\rangle
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e & \to_{\mathsf{E}} & \bar{\mu}[(\sigma_1\, x).\,\langle \sigma_1\, x \,\|\, e \rangle \mid (\sigma_2\, y).\,\langle \sigma_2\, y \,\|\, e \rangle]
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t & \to_{\mathsf{E}} & \mu \star .\, \langle t \,\|\, \star \rangle \\
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η-expansions are perfectly regular.

The structure of good R-normal forms

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$\langle (t, u) \parallel \bar{\mu}x.\, c \rangle$	reducible
$\langle (t, u) \parallel \bar{\mu}[(\sigma_1\, x).\, c_1 \mid (\sigma_2\, y).\, c_2] \rangle$	bad

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$\langle (t, u) \parallel \bar{\mu}x.\, c \rangle$	reducible
$\langle (t, u) \parallel \bar{\mu}[(\sigma_1\, x).\, c_1 \mid (\sigma_2\, y).\, c_2] \rangle$	bad
$\langle (t, u) \parallel \star \rangle$	good (constructor)

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$\langle (t, u) \parallel \bar{\mu}x.\, c \rangle$	reducible
$\langle (t, u) \parallel \bar{\mu}[(\sigma_1\, x).\, c_1 \mid (\sigma_2\, y).\, c_2] \rangle$	bad
$\langle (t, u) \parallel \star \rangle$	good (constructor)
$\langle \mu(x \cdot \star).\, c \parallel \star \rangle$	good (abstractor)

The structure of good R-normal forms

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General form: **phase structure**

$$f ::= \quad \langle x \parallel S[f] \rangle \quad \mid \quad \langle V[f] \parallel \star \rangle$$

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General form: **phase structure**

$$f ::= \quad \langle x \parallel S[f]\rangle \quad \mid \quad \langle V[f] \parallel \star\rangle$$

Modulo E-expansions, we can assume that S or V contain either an abstractor, or only constructors or single-variable $\mu, \bar\mu$.

$$\langle x \parallel V \cdot \pi_1\, \bar\mu(x_1, x_2).\, f\rangle \qquad \to_{\mathsf{E}} \qquad \big\langle x \,\big\|\, V \cdot \pi_1\, \bar\mu y.\, \langle y \parallel \bar\mu(x_1,x_2).\, f\rangle\big\rangle$$

Computing the RE-equivalence of two R-normal forms

$$x : X, f : (X \to Y + Z) \vdash Y + Z$$

$$(f\ x) \qquad\qquad \delta(f\ x,\ y.\sigma_1\ y,\ z.f\ x)$$

Traduction as (R-normal) configurations:

$$\langle f \,\|\, x \cdot \star \rangle \qquad \Big\langle f \,\Big\|\, x \cdot$$

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Step 2: commuting phases up in the term, respecting scope only.

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Computing the equivalence of two normal forms (1, 2, 3)

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Computing the equivalence of two normal forms (soleil)

We are left to compare

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Final step: instead of a type-directed η-expansion, traverse both terms and E-expand on demand (on all abstractors)

$$\frac{?}{c \overset{?}{\simeq} \langle x \,\|\, \bar{\mu}[(\sigma_1\, y_1).\, c_1 \mid (\sigma_2\, y_2).\, c_2] \rangle}$$

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Conclusion

Take away: $\mu\bar{\mu}$ is an abstract-machine calculus with highly regular syntax, reduction/expansion, and equational theory.

Plenty was left under the hood.

$\mu\bar{\mu}$ uses a **polarized** evaluation order, subsuming call-by-name and call-by-value.

$\mu\bar{\mu}$ supports effectful constructors (eg. function call); the polarized R and E-equivalences are weaker than shown here.

We need to explicitly assume purity (commutativity, idempotence, cancellability) to recover full $\beta\eta$.

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Thank you!