Which types have a unique inhabitant? Focusing on pure program equivalence

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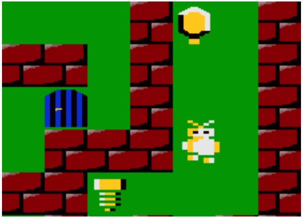
Programming

Program execution		Program description
	$\Longleftarrow$	(code below)
ML Robbo		Tomasz Kokoszka

```
let rec game_loop () =
  let events = Input.process_events () in
  let new_action = Sdltimer.get_ticks () > !last_action + !delay in
  if new_action || events then begin
    last_action := Sdltimer.get_ticks ();
    Level.next_action selected_action;
    (* Odrysowanie *)
    Graph.hide_console ();
    Level.draw_image ();
    Level.draw_info ();
    Console.draw ();
    Video.flip ();
    Gc.major ();
  end else
    Sdltimer.delay 1;
  game_loop ()
```

Like writing cooking recipe for a very very literal cook.

Code inference

Some program fragments are **boring** to write.

Dear computer, please guess what I mean here.
It should be clear from the context.

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http://www.papillesetpupilles.fr/2011/01/crepes-faciles.html

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puis les œufs sans leur coquille

Implicit transformation, coercion: $\texttt{egg} \triangleright (\texttt{white} \times \texttt{yolk})$

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mélangez bien...

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Implicit transformation, coercion: $\mathtt{egg} \triangleright (\mathtt{white} \times \mathtt{yolk})$

mélangez bien avec une cuillère

Implicit parameter.

Unicity

"Guess from the context"

What if there are several possible choices?

Refuse to guess? Ask the programmer?
(Heuristics studied in the literature.)

When is there only one choice?
When are all choices equivalent?

Type systems

Not all sentences are grammatically correct.
Not all grammatical sentences are meaningful:

verser le saladier dans le beurre

There are rules to follow to create well-formed, meaningful programs.

In this talk: simply-typed lambda-calculus.
Very simple, restrictive rules.
Realistic languages use richer systems.

"In this context, are all guesses equivalent?"
$\Longrightarrow$
"Given these type constraints, are all guesses equivalent?"

$$
\begin{array}{lll}
A, B, C, D ::= & & \text{(simple) types} \\
& \mid X, Y, Z & \text{atomic types: } \texttt{white}, \texttt{yolk}, \texttt{sugar}\ldots \\
& \mid A \times B & \text{pairs: } \texttt{white} \times \texttt{yolk} \\
& \mid A + B & \text{sums, choices: } \texttt{oil} + \texttt{butter} \\
& \mid A \rightarrow B & \text{functions: } \texttt{white} \rightarrow \texttt{foam} \\
& \mid \ldots &
\end{array}
$$

$$
\begin{aligned}
\mathtt{kitchen} &: \mathtt{salt} \times \mathtt{sugar} \times \mathtt{oil} \\
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\end{aligned}
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$$\vdash$$

$? : \mathtt{meringue}$

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Two different choices!

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Additional difficulties: equivalent programs, infinite search.

Proofs-programs correspondence (Curry-Howard)

Another domain has to respect strict rules of well-formedness: mathematical proofs.

Types connect programming languages to logic (study of formal proofs).

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Types connect programming languages to logic (study of formal proofs).

Type system	**Logic**
Type	Proposition
Program	Formal proof
$A \times B$	A and B
$A + B$	A or B
$A \rightarrow B$	A implies B
Program search	Proof search

Proofs-programs correspondence (Curry-Howard)

Another domain has to respect strict rules of well-formedness: mathematical proofs.
Types connect programming languages to logic (study of formal proofs).

Type system	**Logic**
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Inhabitation	Provability
Unicity	N/A

Canonical proofs

Proof theory has studied the question of "canonical" proof representations.

Can we adopt these representations for programs?
Do they lose computational information?

Do logical techniques respect program equivalence?

Contribution

A functional programming problem:
Which types have a unique inhabitant?

A method from logic: focusing.
New bridge between the communities.
(Noam, Teyjus, Psyche, L. More elementary, new audience.)

A result: Unique inhabitation is decidable for the simply-typed lambda-calculus with atoms, sums and empty types.
An effective algorithm. For any $(\Gamma \vdash ? : A)$ returns 0, 1 or 2 programs.

Examples

Abstract library functions. Glue code.

$$\texttt{fst} : \alpha \times \beta \to \alpha \qquad \texttt{?} : \alpha \to \alpha \to \alpha$$

$$\texttt{curry} : (\alpha \times \beta \to \gamma) \to \alpha \to \beta \to \gamma$$

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$$\texttt{error}\ A \stackrel{\text{def}}{=} \texttt{exn} + A$$

$$\texttt{Monad error} \quad \begin{cases} \texttt{return} & : \ \alpha \to \texttt{error}\ \alpha \\ \texttt{bind} & : \ \texttt{error}\ \alpha \to (\alpha \to \texttt{error}\ \beta) \to \texttt{error}\ \beta \end{cases}$$

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$\texttt{Functor error}$ (unique)

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`Applicative error` (not unique)

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`Functor error` (unique)

`Applicative error` (not unique)

$$\texttt{ap} : \texttt{error}\ (\alpha \to \beta) \to \texttt{error}\ \alpha \to \texttt{error}\ \beta$$

General idea

Enumerate all possible terms in $(\Gamma \vdash ? : A)$:

$$t_1, t_2, \ldots, t_n$$

Stop if we find two distinct terms.

We must remove duplicates.

$$t_1, (\lambda x.\, x)\ t_1, (\lambda x.\, x)\ ((\lambda x.\, x)\ t_1), \ldots$$

Infinitely many duplicates $\Longrightarrow$ non-termination.

Enumerate terms without duplicate: canonical representations.

β-normal forms

Easy in the **negative** fragment $(\to, \times, 1)$. Values and neutrals.

$$
\begin{aligned}
v ::= &\ \text{values} \\
&\mid \lambda x.\, v \\
&\mid (v_1, v_2) \\
&\mid n
\end{aligned}
\qquad\qquad
\begin{aligned}
n ::= &\ \text{neutrals} \\
&\mid x \\
&\mid \pi_i\ n \\
&\mid n\ v
\end{aligned}
$$

Problem: no clear way to add **positives** $(+, 0)$.

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$$v, w ::= \text{values} \mid \lambda x.\, v \mid (v_1, v_2) \mid n \mid\ ?\ \sigma_i\ v$$

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$$?\ \mid \texttt{match}\ n\ \texttt{with}\ \mid \sigma_1\ x_1 \to v_1 \mid \sigma_2\ x_2 \to v_2$$
$$?\ \mid \texttt{match}\ n\ \texttt{with}\ \mid \sigma_1\ x_1 \to n_1 \mid \sigma_2\ x_2 \to n_2$$

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\begin{array}{rcl}
v & ::= & \text{values} \\
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  & \mid & (v_1, v_2) \\
  & \mid & n
\end{array}
\qquad
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\end{array}
$$

$$
\left(\texttt{match}\ n\ \texttt{with} \;\middle|\; \begin{array}{l} \sigma_1\ x_1 \rightarrow \lambda y.\, v_1 \\ \sigma_2\ x_2 \rightarrow \lambda y.\, v_2 \end{array} \right) w
$$

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Easy in the **negative** fragment $(\to, \times, 1)$. Values and neutrals.

$$v ::= \text{values} \mid \lambda x.\, v \mid (v_1, v_2) \mid n \qquad n ::= \text{neutrals} \mid x \mid \pi_i\ n \mid n\ v$$

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Focusing (overview)

$$\frac{}{x : X + Y \vdash Y + X} \quad ?$$

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$$\dfrac{\overline{x : X + Y \vdash Y}}{x : X + Y \vdash Y + X} \qquad \sigma_1 \; ?$$

Focusing (overview)

$$\dfrac{\dfrac{x_1 : X \vdash Y \qquad x_2 : Y \vdash Y}{x : X + Y \vdash Y}}{x : X + Y \vdash Y + X}$$

$$\sigma_1 \left(\texttt{match } x \texttt{ with} \left| \begin{array}{l} \sigma_1\, x_1 \to ? \\ \sigma_2\, x_2 \to ? \end{array} \right. \right)$$

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non-invertible rule: risky **choice**. σ_i ...

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Focusing imposes restrictions based on invertibility.

Focused normal forms (Overview)

$$\begin{aligned} v ::= &\ \text{values} \\ &\mid \lambda x.\, v \\ &\mid (v_1, v_2) \\ &\mid n \end{aligned} \qquad \begin{aligned} n ::= &\ \text{neutrals} \\ &\mid x \\ &\mid \pi_i\ n \\ &\mid n\ v \end{aligned}$$

Remark that values are **invertible**, neutrals **non-invertible**.

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(The details are a bit different.)

Focusing is not enough

For

$$x : X + Y \vdash ? : Y + X$$

there is a unique focused normal form (good!).

But not for

$$f : 1 \to (X + Y) \vdash ? : Y + X$$

Applying a function is non-invertible, no imposed ordering.

Can be applied:

- before or after making choices
- zero, one, many times

Saturation

Focusing forces splitting of **values** of sum type.
Computations of sum type may be done at any point.
Duplicates (in a pure setting).

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Run computations of sum type as **early** as possible.

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Duplicates (in a pure setting).

Saturation: deduce **all** positives before making any choice.
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Stronger than focusing: no more duplicates.

Demo time

Implementation available:
`https://gitlab.com/gasche/unique-inhabitant`

$$f : (1 \to X + Y)$$

$$\vdash$$

$$? : Y + X$$

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$$f : (1 \to X + Y)$$

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$$\texttt{let } z^{X+Y} = f\ ()\ \texttt{in } ? : Y + X$$

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Final result: zero, one or two (distinct) terms.

Two-or-more approximation

Does saturation terminate in general?

$$f : \mathbb{N} \to Y + Y, \ldots$$

Even when they are infinitely many programs, **proof** search terminates.

Subformula property: normal forms use finitely many types/formulas.

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Result: to detect unicity, keep at most **two** variables of each type.

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$$x_1 : X, x_2 : X, x_3 : X, \ldots$$

Result: to detect unicity, keep at most **two** variables of each type.

Consequence: Saturation can stop after **two** proofs of each positive:

$$0, 1, \text{many}$$

Sequent calculus (logic)

$$\frac{\Gamma \vdash A \qquad \Gamma, B \vdash C}{\Gamma, A \to B \vdash C} -$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}$$

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$$\frac{\Gamma, A_i \vdash C}{\Gamma, A_1 \times A_2 \vdash C}$$

$$\frac{\Gamma \vdash A_1 \qquad \Gamma \vdash A_2}{\Gamma \vdash A_1 \times A_2}$$

$$\frac{}{\Gamma \vdash 1}$$

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$$\frac{\Gamma \vdash A_1 \qquad \Gamma \vdash A_2}{\Gamma \vdash A_1 \times A_2}$$

$$\frac{}{\Gamma \vdash 1}$$

$$\frac{\Gamma, A_1 \vdash C \qquad \Gamma, A_2 \vdash C}{\Gamma, A_1 + A_2 \vdash C}$$

$$\frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 + A_2} +$$

$$\frac{}{\Gamma, 0 \vdash A}$$

Sequent calculus (logic)

$$\frac{\Gamma \vdash A \qquad \Gamma, B \vdash C}{\Gamma, A \to B \vdash C} -$$

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$$\frac{\Gamma, A_i \vdash C}{\Gamma, A_1 \times A_2 \vdash C} -$$

$$\frac{\Gamma \vdash A_1 \qquad \Gamma \vdash A_2}{\Gamma \vdash A_1 \times A_2}$$

$$\frac{}{\Gamma \vdash 1}$$

$$\frac{\Gamma, A_1 \vdash C \qquad \Gamma, A_2 \vdash C}{\Gamma, A_1 + A_2 \vdash C}$$

$$\frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 + A_2} +$$

$$\frac{}{\Gamma, 0 \vdash A}$$

Negatives: $(\to, \times, 1)$

Positives: $(+, 0)$

Focusing

Invertible phase: apply invertible rules as early as possible.

$$\Gamma \vdash_{\mathsf{inv}} A$$

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When all invertible rules are done, we must **choose** a focus.

$$\Gamma \vdash_{\mathsf{foc}} A$$

(Γ negative, A positive)

Focusing

Invertible phase: apply invertible rules as early as possible.

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When all invertible rules are done, we must **choose** a focus.

$$\Gamma \vdash_{\mathsf{foc}} A$$

(Γ negative, A positive)

Focus on a formula:

- on the right: $\Gamma \vdash_{\mathsf{foc.r}} [A]$
- or on the left: $\Gamma, [A] \vdash_{\mathsf{foc.l}} B$

Then keep applying non-invertible rules on the formula under focus.

Focused λ-calculus

$$
\begin{aligned}
t, u, r ::= \quad & \textit{invertible} \\
\mid\ & \lambda x.\, t \\
\mid\ & (t, u) \\
\mid\ & \texttt{match } x \texttt{ with} \left| \begin{array}{l} \sigma_1\, x \to u_1 \\ \sigma_2\, x \to u_2 \end{array} \right. \\
\mid\ & () \\
\mid\ & \texttt{absurd}(x) \\
\mid\ & (f : A_{pa})
\end{aligned}
$$

$$
\begin{aligned}
f, g ::= \quad & \textit{choice of focus} \\
\mid\ & (n : X) \\
\mid\ & \texttt{let } (x : A_p) = n \texttt{ in } t \\
\mid\ & (p : A_p)
\end{aligned}
$$

$$
\begin{aligned}
n, m ::= \quad & \textit{negative neutral} \\
\mid\ & \pi_i\, n \\
\mid\ & n\, p \\
\mid\ & (x : A_n)
\end{aligned}
$$

$$
\begin{aligned}
p, q ::= \quad & \textit{positive neutral} \\
\mid\ & \sigma_i\, p \\
\mid\ & (x : X) \\
\mid\ & (t : A_{na})
\end{aligned}
$$

Saturation

$$\frac{\Gamma, \Delta \vdash_{\mathsf{foc}} A}{\Gamma; \Delta \vdash_{\mathsf{inv}} A}$$

Old and New.

Saturation

$$\frac{\Gamma, \Delta \vdash_{\text{foc}} A}{\Gamma; \Delta \vdash_{\text{inv}} A}$$

Old and New.

Left focusing: $\Gamma_{na} \vdash \texttt{let}\ (x : A_p) = n\ \texttt{in}\ t$

Saturation: $\Gamma; \Delta \vdash \texttt{let}\ (x_i)^{i \in I} = (n_i)^{i \in I}\ \texttt{in}\ t$

$$\text{where}\ (n_i)^{i \in I} \stackrel{\text{def}}{=} \{\ \Gamma, \Delta \vdash n \Downarrow A_p \ \mid\ n \text{ uses } \Delta\ \}_2$$

Saturation

$$\frac{\Gamma, \Delta \vdash_{\text{foc}} A}{\Gamma; \Delta \vdash_{\text{inv}} A}$$

Old and New.

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All typed terms: $\Gamma \vdash t : A$

Focused terms: $\Gamma^{\text{at}} \vdash_{\text{foc}} f : P^{\text{at}}$

Saturated terms: $\Gamma^{\text{at}}; \Gamma^{\text{at}\prime} \vdash_{\text{sat}} g : P^{\text{at}}$

Canonicity

Completeness

Focusing: (known result) $\Gamma^{\mathsf{at}} \vdash_{\mathsf{foc}} t \rightsquigarrow f : P^{\mathsf{at}}$ with $t \approx_{\beta\eta} f$

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Theorem (Canonicity)

$$\left.\begin{array}{l} \Gamma^{\mathsf{at}} \vdash_{\mathsf{foc}} f_1 \approx_{\beta\eta} f_2 : P^{\mathsf{at}} \\ \emptyset; \Gamma^{\mathsf{at}} \vdash_{\mathsf{sat}} f_1 \rightsquigarrow g_1 : P^{\mathsf{at}} \\ \emptyset; \Gamma^{\mathsf{at}} \vdash_{\mathsf{sat}} f_2 \rightsquigarrow g_2 : P^{\mathsf{at}} \end{array}\right\} \implies g_1 = g_2$$

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Result: searching among saturated terms decides unicity

Corollary: equivalence with sums and empty type is decidable

Future work

Semantic proof of equivalence with sums and empty type.

$$g_1 = g_2 \qquad \Rightarrow \qquad g_1 \approx_{\beta\eta} g_2 \qquad \Rightarrow \qquad g_1 \approx_{\texttt{ctx}} g_2$$

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Unicity modulo equations?

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Purity and types are not just restrictions
they can make programming **easier**.