Multi-focusing on extensional rewriting with sums

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Abstract—We propose a logical justification for the rewriting-based equivalence procedure for simply-typed lambda-terms with sums of Lindley [Lin07]. It relies on maximally multi-focused proofs, a notion of canonical derivations introduced for linear logic. Lindley’s rewriting closely corresponds to preemptive rewriting [CMS08], a technical device used in the meta-theory of maximal multi-focus.

This is the expanded version of an article of the same name that has been submitted for review.

I. INTRODUCTION

Deciding observational equality of pure typed lambda-terms in presence of sum types is a difficult problem. After several solutions based on complex syntactic or semantic techniques, Sam Lindley presented a surprisingly simple rewriting solution [Lin07]. While the underlying intuition (extrude contexts to move pattern-matchings as high as possible in the term) makes sense, we may still wonder exactly how ad-hoc the solution is, and why the previous highly-principled approach did not discover this relieving presentation.

In this paper, we will propose a logical justification of this algorithm. It is based on recent developments in proof search, maximally multi-focused proofs [CMS08]. The notion of preemptive rewriting was introduced in the meta-theory of multi-focusing as a purely technical device; we claim that it is in fact strongly related to Lindley’s rewriting, and formally establish the correspondence.

The reference work on multi-focused systems [CMS08] has been carried in a sequent calculus for linear logic. We will first establish the meta-theory of maximal multi-focusing for intuitionistic logic (Section II). We start from a sequent calculus presentation, which is closest to the original system [CMS08]. Our first contribution is to propose an equivalent multi-focusing system in natural deduction [I-B]. We then define preemptive rewriting in this natural deduction [I-L] and establish canonicity of maximally multi-focused proofs [I-F].

In Section III, we transpose the preemptive rewrite rules into a relation on proof terms. This allows us to precisely study the correspondence between rewriting a multi-focused proof into a canonical maximally multi-focused one, and Lindley’s \(\gamma\)-reduction on lambda-terms. We demonstrate that they compute the same normal forms, modulo a notion of redundancy elimination that is missing in the multi-focused system.

We finally introduce redundancy-elimination rewriting and equivalence for the proof terms of the multi-focused natural deduction (Section IV). The resulting notion of canonical proofs, simplified maximal proofs, precisely corresponds to normal forms of Lindley’s rewriting relation. The natural notion of local equivalence between simplified maximal proofs therefore captures extensional equality.

II. INTUITIONISTIC MULTI-FOCUSING

The space of proofs in sequent calculus or natural deduction exhibits a lot of redundancy: many proofs that are syntactically distinct really encode the same semantics. In particular, it is often possible to permute two inference rules in a way that preserves the validity of proofs, but also the reduction semantics of the corresponding proof terms. If a permutation transforms a proof with rule \(A\) applied above rule \(B\) into a proof with rule \(B\) applied above rule \(A\), we say that it is an \(A/B\) permutation (\(A\) is above the slash, as in the source proof).

Focusing is a general discipline that can be imposed upon proof system, based on the separation of inference rules into two classes. Inverse rules (called as such because their inverse is admissible) always preserve provability, and can thus be applied as early as possible. Non-inverse rules may result in dead ends if they are applied too early (consider proving \(A+B\) by introducing the sum on the right-hand side). In focusing calculi, derivations are structured in “sequences” or “phases”, that either only apply inverse rules or only non-inverse rules. Focusing imposes that phases be as long as possible. During inverse phases, one must apply any valid inverse rule. During non-inverse phases, one focuses on a set of formulas, and applies non-inverse operations on those formulas as long as possible – if the phase is started too early, this may result in a dead end.

Inversibility is related to the notion of polarity of logical connectives: we call positive those whose right-hand-side rule is non-inversible (they are “only interesting in positive position”), and negative those whose left-hand-side rule is non-inversible. In single-succedent intuitionistic logic, \((\rightarrow)\) is negative, + is positive, and * may actually be assigned either polarity – we consider it negative in the present work. Atomic formulas may be assigned polarities as well, but we will mostly ignore this aspect which is orthogonal to our work.

In single-sided calculi, non-inverse rules are those that introduce positive connectives, and are called “positive”. For continuity of vocabulary, we will also call non-inverse rules positive, and inverse rules negative. In particular, a permutation that moves a non-inverse rule below an inverse rule is a “pos/neg permutation”.

A. Multi-focused sequent calculus

Multi-focusing ([MS07], [CMS08]) is an extension of focusing calculi where, instead of focusing on a single formula
of the sequent (either on the left or on the right), we allow to simultaneously focus on several formulas at once. The multiple foci do not interact during the focusing phase, and this allows to express the fact that several focusing sequences are in fact independent and can be performed in parallel, condensing several distinct focused proofs into a single multi-focused derivation.

We start with a multi-focused variant of the intuitionistic sequent calculus, presented in Fig. 1. We denote focus using brackets, but this notation will change in natural deduction calculus; others syntactic conventions will be preserved throughout the paper.

In particular, we write $A_n$ or $\Delta_p$ for formula or contexts that must be all negative or positive, and $X$, $Y$ or $Z$ for atoms (we do not use lowercase to avoid future confusions with proof terms). For readability reasons, we only add polarity annotations when absolutely necessary; if we consider only derivations whose end conclusion is unfocused, then the invariant holds that the unfocused left-hand-side context is always all-negative, while the unfocused right-hand-side formula is always positive (when present).

Our intuitionistic calculi are, as is most frequent, single-succedent. The notation $A, B$ on the right does not denote a real disjunction but a single formula, one of the two variables being empty. The focusing rule with conclusion $\Gamma, \Delta \vdash A, B$ can be instantiated in (at least) two situations, one where $A$ is empty, and the premiss is $\Gamma, [\Delta] \vdash [B]$ (the succedent is part of the multi-focus), and one where $B$ is empty, and the premiss is $\Gamma, [\Delta] \vdash A$ (the succedent is not part of the multi-focus). Note that $\Delta$ is a set and may be empty, in which case the focus only happens on the right.

A minor presentation difference to the reference work on multi-focusing [CMS08] is that all the formulas under focus are released at the same time.

This multi-focused calculus proves exactly the same formulas as the singly-focused sequent calculus. The latter is trivially included in the former, and conversely one can turn a multi-focus into an arbitrarily ordered sequence of single foci. As a corollary, relying on non-trivial proofs from the literature (e.g., [Sim11]), it is equivalent in provability to the (non-focused) sequent calculus for intuitionistic logic.

B. Multi-focused natural deduction

While the multi-focusing sequent calculus closely corresponds to existing focused presentations, its natural presentation in Fig. 2 is new. We took inspiration from the presentation of focused linear logic in natural deduction of [BNS10], in particular the $\uparrow$ and $\downarrow$ notations coming from intercalation calculi.

There are three judgments, $\Gamma \vdash A$ is the unfocused judgment with the invertible rules. $\Gamma \uparrow \Delta$ is the “elimination-focused” judgment (focused on $\Delta$), and $A \uparrow \uparrow B$ is the “introduction-focused” judgment (focused on $A$). The two focused judgments do not come separately (any multi-focus can possibly have both left and right foci), but are introduced by the same focusing rule.

The multi-focusing rule focuses on a (possibly empty) set of formula $\Delta$ that result from non-invertible eliminations rules, and optionally on a formula $A$ that will be the principal formula of non-invertible introduction rules. The $A \uparrow \uparrow B$ judgment means that either $A \uparrow B$ ($A$, under focus, is refined into $B$) or that $A = B$ – there is no focus on the right in this multi-focusing.

Finally, in the multi-focusing rule, the negativity restriction on the context $\Delta$, and the positivity restriction on the formula $B$, guarantee that non-invertible rules are applied as far as possible – giving synthetic positive connectives, a central aspect of focusing.

Lemma 1: The multi-focused natural deduction system proves exactly the same non-focused judgments as the multi-focused sequent calculus.

Proof: Our proof is similar in spirit to the usual proofs of equivalence between sequent calculus and natural deduction: reversing a sequence of left-focusing rules in sequent calculus upside-down gives a sequence of elimination-focused rules in natural deduction – and conversely.

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Fig. 1. Multifocused sequent calculus for intuitionistic logic

\[
\begin{array}{c}
\Gamma, \Delta \vdash A, B \\
\downarrow \Delta \vdash \Delta, A \\
\Gamma \vdash \Delta \vdash [A_i] \\
\Gamma \vdash A, [\Delta] \vdash [A_1 + A_2] \\
\end{array}
\]

Fig. 2. Multifocused natural deduction for intuitionistic logic
For an example, consider the following proof:

\[
\begin{align*}
S \quad \Gamma, [\Delta] \vdash [B_1] & \iff \left( \text{elim}(S, \Delta), \frac{B_1 \triangleright B'}{B_1 + B_2 \triangleright B'} \right) \\
S \quad \Gamma, [\Delta', C_1] \vdash [A, [B]] & \iff \left( \Delta \triangleright \Delta', C_{1} \triangleright C_{2} \quad \Delta \triangleright \Delta, C_{1} \quad \frac{\text{intro}(S, B')}{{\text{elim}(S, \Delta)}} \right) \\
S \quad \Gamma, [\Delta', C_1 \triangleright C_2] \vdash [A, [B]] & \iff \left( \Delta \triangleright \Delta', C_{1} \triangleright C_{2} \quad \Delta \triangleright \Delta, C_{1} \quad \frac{\text{intro}(S, B')}{\text{elim}(S, \Delta)} \right)
\end{align*}
\]

It gets reversed into the corresponding natural deduction derivation:

\[
\begin{align*}
\frac{(X \ast Y) \ast Z, X \vdash X}{}
\frac{(X \ast Y) \ast Z, [(X \ast Y) \ast Z] \vdash X}{}
\frac{(X \ast Y) \ast Z, [(X \ast Y) \ast Z] \vdash X}{}
\frac{(X \ast Y) \ast Z \vdash X}{}
\end{align*}
\]

We show how to map each fragment of a derivation starting with a focusing rule and ending with release rules – in fact exactly one release rule – from one system to another. This is sufficient, as we can then rewrite whole derivations by structural induction from the leaves to the root.

Consider a focused sequence \( S \) surrounded by a multi-focusing and a release rule

\[
\begin{align*}
\pi & \quad \frac{\Delta_{n} \triangleright \Delta_{n'}}{\Gamma_{n}, \Delta_{n'} \vdash A_{p}, B_{n'}} \\
\pi & \quad \frac{\Gamma_{n}, [\Delta_{n}] \vdash A_{p}, [B_{n}]}{\Gamma_{n}, [\Delta_{n}] \vdash A_{p}, [B_{n}]} \\
S & \quad \frac{\Gamma_{n}, \Delta_{n} \vdash A_{p}, [B_{n}]}{\Gamma_{n}, \Delta_{n} \vdash A_{p}, [B_{n}]} \\
\end{align*}
\]

We rewrite it into the following derivation fragment in natural deduction (with some context weakening applied to highlight that it would be valid in linear natural deduction as well):

\[
\begin{align*}
\frac{\Delta_{n} \triangleright \Delta_{n'}}{\Gamma_{n}, \Delta_{n} \vdash A_{p}, B_{n'}} \\
\frac{\text{intro}(S, B')}{\text{elim}(S, \Delta)} \quad \pi & \quad \frac{B_{n'} \triangleright B'}{B_{n'} \triangleright B'} \\
\frac{\Gamma_{n}, \Delta_{n} \vdash A_{p}, B_{n'}}{\Gamma_{n}, \Delta_{n} \vdash A_{p}, B_{n'}} \\
\end{align*}
\]

Where \( \text{elim}(S, \Delta) \) and \( \text{intro}(S, B') \) are sequences of focusing rules defined by induction on \( S \):

\[
\begin{align*}
\text{elim}(S, \Delta) & \quad \Delta_{n} \triangleright \Delta_{n'} \quad \frac{B_{n'} \triangleright B'}{B_{n'} \triangleright B'} \\
\text{intro}(S, B') & \quad \frac{\Gamma_{n}, \Delta_{n} \vdash A_{p}, B_{n'}}{\Gamma_{n}, \Delta_{n} \vdash A_{p}, B_{n'}}
\end{align*}
\]

Note that applying an inference rule on top of \( \text{elim}(S, \Delta) \) is well-defined because elimination-focused rules always have exactly one elimination-focused premise, which we will call its top judgment.

The fragments of derivation built are well-formed thanks to the following invariants:

- (when it is non-empty) the focused right formula of the bottom judgment of \( \text{intro}(S, B') \) is the same as the one of the bottom judgment of \( S \)
- the focused context of the top focused judgment of \( \text{elim}(S, \Delta) \) is the same as the focused context of the bottom judgment of \( S \)

Remark that the mapping we have defined is bijective on the top and bottom judgments of the focused derivation fragments – it preserves provability back and forth – but not on the derivation themselves. Indeed, several distinct focusing fragments of the sequent calculus may map to the same natural deduction fragment, if they only differ in the ordering of left- and right-introduction rules. We recover the well-known fact that natural deduction (and lambda-calculus as its term language) has less bureaucracy than sequent calculus. The sequent calculus allows for unsignificant ordering that must be eliminated by quotienting over pos/pos permutations, or fixing a forced order; whereas the syntax of natural deduction exhibits a spatial parallelism between the introduction and elimination rules of a focusing sequence that structurally removes (some of the) redundancy. We re-introduced some redundancy by allowing parallel eliminations to be interleaved in the elimination-focusing rules, but that is not a concern as we decided to quotient derivations over local permutations in any case.

\[\square\]

C. A preemptive variant of multi-focused natural deduction

Multi-focusing was introduced to express the idea of parallelism between non-inversible rules on several independent
A proof has more parallelism than another if two sequential foci of the latter are merged (through rule permutations) in a single multi-focus in the former. A natural question is whether there exists “maximally parallel proofs”. To answer it (affirmatively), the original article on multi-focusing \cite{CMS08} introduced a rewriting relation that permutes non-inversible phases down in proof derivations, until they cannot go any further without losing provability – neighbouring phases can then be merged into a maximally focused proof.

In the process of moving down, a non-inversible phase will traverse irreversible phases below. The intermediary states of this reduction sequence may break the invariant that irreversible rules must be applied as early as possible; we say that the non-inversible phase preempts (a part of) the inversible phase. As this intermediary state is not a valid proof in off-the-shelf multi-focusing systems, the original article introduced a relaxed variant called a preemptive system, in which the phase-sinking transformation, called preemptive rewriting, can be defined.

The original preemptive for linear logic \cite{CMS08} is a

\begin{center}
\begin{figure}
\centering
\begin{align*}
\Gamma, A \vdash B & \quad \Gamma, A' \vdash B' \quad \Gamma, A, A' \vdash B \\
\Gamma, A \vdash B & \quad \Gamma, A', A'' \vdash B' \quad \Gamma, A, A', A'' \vdash B \\
\Gamma, A \vdash B & \quad \Gamma, A', A'' \vdash B' \quad \Gamma, A, A', A'' \vdash B \\
\Gamma, A \vdash B & \quad \Gamma, A', A'' \vdash B' \quad \Gamma, A, A', A'' \vdash B \\
\end{align*}
\end{figure}
\end{center}
variant of the multi-focused linear logic which has single-sided non-invertible judgments of the form $\Gamma \downarrow \Delta$. It adds a specific position in the sequent for suspended invertible formulas – the $\Xi$ in $\vdash \Gamma \downarrow \Delta; \Xi$.

We present in Fig. 4 a preemptive calculus for multi-focused natural deduction, which adopts a different convention to have lighter notations. We allow the non-focused context to contain invertible formulas, writing the intuitionistic equivalent of $\vdash \Gamma, \Xi \downarrow \Delta$. The focusing rule can then be made more liberal to allow preemption – i.e., the start of a focusing phase when invertible rules could still be applied.

The elimination-focused judgment $\Gamma \uparrow \Delta$ still needs to be modified to express the fact that some focusing present at its leaves, of the form $\Gamma \uparrow \Delta'$, may preempt the current focusing phase to be move under it in the derivation; this suggests a judgment of the form $\Gamma; \Delta', \Delta' \downarrow \Delta$.

The rules are presented in Fig. 4 except for the invertible and focused-introduction rules that are strictly unchanged from the previous multi-focused rules in Fig. 2. Any derivation of the previous system can trivially be seen as a derivation of the current one, with empty suspended contexts in all elimination-focused rules.

Note that the context $\Gamma'$ suspended in the elimination-focused rules is made available to the unfocused premise of arrow elimination. When a focusing phase of the unfocused premise is moved down the derivation during preemption, it still needs to be available to the premise. It is thus placed in the suspended context. This is the only purpose of this suspended context: pair elimination passes it unchanged and the release rule ignores it.

We needed an extra component in this judgment, as neither adding $\Gamma'$ to the unfocused context $\Gamma$ or to the focused context $\Delta$ would be satisfying. Adding it to $\Delta$ does not work, as it would not be available in the unfocused premise of arrow elimination (allowing $\Delta$, which may be a not-yet-finished focusing sequence, would break the focusing discipline). Adding it to $\Gamma$ is unpleasant, as it would be able to influence the focused context $\Delta_n$ in the release rule, breaking the independence of suspended context from the current focus – which is not strictly necessary but simplifies reasoning on proofs.

Preempting the invertible rules is made possible by the relaxed focusing rule, which is unchanged except that the context $\Gamma_{np}$ is allowed to have both negative and positive formulas – the notation is only meant to emphasize this. Preempting non-invertible rules with another focusing sequence is allowed by a new rule for elimination-focused judgment. We would also need one for introduction-focused judgment, if we had a positive connective with an unfocused premise in its introduction rule.

### D. Preemptive rewriting

We can then define in Fig. 3 the rewriting relation on the preemptive calculus, that lets any phase of negative focus move as far as possible down the derivation tree. Maximally multi-focused proofs, which can be characterized on permutation-equivalence classes of multi-focused proofs, are also the normal forms of this rewriting relation.

A focused phase cannot move below an inference rule if some of the foci depend on this inference rule. Instead of expressing the non-dependency requirement by implicit absence of the foci, we have explicitly cancelled out the foci that must be absent to improve readability. For example, you cannot move a right-focus on $A$ below the split of $A * B$.

In this situation, it may be the case that others part of the multi-focus do not depend on the rule below, and those should not be blocked. To allow rewriting to continue, the last rewrite of our system is bidirectional. It allows to separate the foci of a multi-focus, in particular separate the foci that depend on the rule below from those that do not – and can thus permute again. This correspond to the first rule of the original preemptive rewriting system [CMS08], which splits a multi-focus in two.

In the left-to-right direction, this rule relies on the possibility of merging together two elimination-focused derivations, or two optional introduction-focused derivations, with the implicit requirement that at least one of them is empty.

Finally, we have sometimes used weakening on the right-hand-side of the rewrite rules to emphasize that judgments (and thus provability) are preserved. This also lets us remark that those rewrite do not fundamentally rely on duplication, they could be expressed in a linear system or any system with multiplicative rules.

### E. Reinversion

After the preemptive rewriting rules have been applied, the result is not, in general, a valid derivation in the non-preemptive system. Consider for example the following rewriting process:
We are here representing derivations from a high-level point of view, by naming complete sequences of rules of the same polarity. Sequences of positive (non-invertible) are named \(\pi_n\), and sequences of negative (invertible) rules \(\nu_m\). We use horizontal position to denote parallelism, or dependencies between phases: each dipole \((\pi_k,\nu_k)\) is vertically aligned as the invertibles of \(\nu_k\) have been produced by the foci of \(\pi_k\), but we furthermore assume that the second dipole depends on formulas released by the first, while the third dipole is independent.

The third dipole is independent from the others, and its foci in \(\pi_3\) move downward in the derivation as expected in the preemptive system. After the first step, its negative phase has preempted the invertible phase \(\nu_2\), and it is thus written \(\pi_3; \nu_2\) to emphasize that any rule of this sequence will have all the invertible formulas of \(\nu_2\) in non-focused positions (positives in the hypotheses, and negatives in the succedent). It can then be merged with the foci of \(\pi_2\), in which case it does not see the invertibles of \(\nu_2\) anymore. When it moves further down, the invertible formulas in its topmost sequent, those consumed by \(\nu_3\), are present/preempted by all the non-invertible rules of \(\pi_2\). It is eventually merged with \(\pi_1\).

The normal form of this rewrite sequence could be considered a maximally multi-focused proof, in the sense that the foci happen as soon as possible in the derivation – which was not the case in the initial proof, where \(\pi_3\) was delayed. However, while the initial proof is a valid proof in the non-preemptive system, the last derivation is not: the invertible formulas produced by \(\pi_3\) are not consumed as easily as possible, but only at the very end of the derivation, and the foci of \(\pi_2\) therefore happen while there are still invertible rules to be applied.

We introduce a reinversion relation between proofs, written \(\mathcal{D} \triangleright \mathcal{E}\), that turns the proof \(\mathcal{D}\) with possible preemption into a proof \(\mathcal{E}\) valid in the non-preemptive system, by doing the inversions where they are required, without changing the structure of the negative phases – the foci are exactly the same. In our example, we have:

\[
\begin{bmatrix}
\nu_2 \\
\nu_3 \\
\pi_3; \nu_2 \\
\pi_1 \\
\end{bmatrix}
\triangleright *
\begin{bmatrix}
\nu_2 \\
\nu_3 \\
\pi_3; \nu_2 \\
\pi_1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\nu_2 \\
\nu_3 \\
\pi_3; \nu_2 \\
\pi_1 \\
\end{bmatrix}
\triangleright *
\begin{bmatrix}
\nu_2 \\
\nu_3 \\
\pi_3; \nu_2 \\
\pi_1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\nu_2 \\
\nu_3 \\
\pi_3; \nu_2 \\
\pi_1 \\
\end{bmatrix}
\rightarrow *
\begin{bmatrix}
\nu_2 \\
\nu_3 \\
\pi_3; \nu_2 \\
\pi_1 \\
\end{bmatrix}
\]

Definition 1 (Rewriting relation): If \(\mathcal{D}\) and \(\mathcal{E}\) are proofs of the non-preemptive system, we write \(\mathcal{D} \triangleright \mathcal{E}\) if there exists a \(\mathcal{E}'\) such that \(\mathcal{D} \triangleright ^* \mathcal{E}' \triangleright \mathcal{E}\).

There are several different ways to define reinversion, which we will detail below.

a) Reinversion as a rewriting relation: We can define reinversion by permuting some invertible phases downward in the derivation. We know this is always possible as neg/neg and neg/pos permutations, those where an invertible rule is originally at the top, always preserve provability. Inversibility properties are clearly explained in Alexis Saurin’s thesis.

We will show (Corollary 1) that global equivalence, that is equivalence modulo arbitrary rule permutations, can be decided by complete preemptive rewriting followed by reinversion. This is a sequentialization of permutations (after quotienting on “local” pos/pos and neg/pos permutations): instead of an arbitrary sequence of permutations, we can just perform a series of pos/neg permutations (preemptive rewriting), followed by a series of neg/pos permutations (reinversion).

While rewriting is the simplest way to define reinversion, we will refrain from defining it explicitly now as it is quite verbose, and less interesting than the preemptive rewriting rules. The rewrite steps will be written in full in the section III-A on proof terms for multi-focusing, where they can be more concisely expressed.

b) Reinversion as partial evaluation: Instead of moving some invertible phases downwards in the derivation, we could traverse it from the root to the leaves. Each time we encounter a positive phase \(\pi; \nu\) that preempts some invertible formulas, we could change it to first perform the necessary invertible phase, then \(\pi\). We can do that without looking at \(\nu\), the actual positive phase that was preempted, and is above in the derivation, as invertibility guarantees that any application order will be equivalent – this is very different from the first point of view, although equivalent modulo permutation of invertible rules.

In each leaf of the invertible phase, we can then add the negative phase \(\pi\) and continue traversing the original derivation. When we encounter the original phase \(\nu\), we should simplify it, according on the branch we are in. For example, if one of the invertible rules was an elimination of \(A + B\), this elimination has been reenacted below \(\pi\); were are either in the \(A\) or the \(B\) branch and can remove the sum-elimination accordingly.

It is very easy to define this transformation formally by relying on a cut-elimination result. Consider the following derivation, preempting the non-invertible phase \(\pi\) over the invertible hypothesis \(A + B\):

\[
S \quad \frac{}{\pi; (A + B) \quad \Gamma_n, A + B \vdash C}
\]

Assuming the admissibility of a cut rule, we can transform
it into:

\[
\begin{array}{c}
\text{CUT} \\
\vdots \\
S
\\ \frac{\pi;(A+B)}{A \vdash A+B} \\
\Gamma_n, A+B \vdash C
\\ \frac{\pi;(A+B)}{B \vdash A+B} \\
\Gamma_n, A+B \vdash C
\\ \frac{\pi;(A+B)}{\Gamma_n, A \vdash C} \\
\Gamma_n, A+B \vdash C
\end{array}
\]

More generally, for any proof \( S \) against a positive context \( \Sigma_p \), we can first perform an identity-expansion of \( \Sigma_p \) and, at each leaf, cut against \( S \). The cut-elimination process then removes any invertible rules against \( \Sigma \) in \( S \). If a negative is the succedent, we can similarly \( \eta \)-expand then cut.

Reinversion corresponds to performing that transformation, on all invertible formulas of the sequents before any multi-focusing rule of the input derivation.

c) Reinversion as completeness of focusing: In Alexis Saurin’s PhD thesis [Sau08], the reinversion operation is somewhat hidden in the proof that two multi-focusing can be merged by moving non-invertible phases downward (Theorem 10.16, p. 234): it simply mentions that reconstructing a valid multi-focused proof is exactly the same technique as the one used to establish the completeness of (single)-focusing in the first place.

In the present work, we are interested in the computational meaning of reinversion, to be able to see where it fits in Lindley’s algorithm. We therefore present it as an explicit computation, instead of the implicit application of a proof.

F. Maximal multi-focusing and canonicity

Now that we have defined the focusing-lowering operation between non-preemptive proof, we present the notion of maximal multi-focusing and its meta-theory: while it can be given a definition without resorting to preemption, it is characterized by the normal forms of the \( \Rightarrow \) relation.

Definition 2: We define \( \text{roots}(D) \), where \( D \) is a derivation in the non-preemptive multi-focusing calculus, as the set of formula over focus in its lowest multi-focusing rule, or \( \emptyset \) if it has none.

Definition 3: We say that two proofs \( D \) and \( E \) are locally equivalent, or iso-polar, written \( D \approx_{\text{loc}} E \), if one can be rewritten into the other using only local positive/positive and negative/negative permutations, preserving their initial sequents.

Definition 4: We say that two proofs \( D \) and \( E \) are globally equivalent, or iso-initial, written \( D \approx_{\text{glob}} E \), when one can be rewritten into the other using local permutations of any polarity (so when seen as proofs in a non-focused system), preserving their initial sequents.

Definition 5: A derivation \( D \) is maximally multi-focused, or maximal, if, for any subderivation \( D' \) of \( D \), for any proof \( E' \approx_{\text{glob}} D' \) we have \( \text{roots}(E') \subseteq \text{roots}(D') \).

Lemma 2 (Soundness and monotonicity of preemptive rewriting): If \( D \Rightarrow E \) then \( D \approx_{\text{glob}} E \) and \( \text{roots}(D) \supseteq \text{roots}(E) \).

Proof: Immediate, by inspection of each rewrite rule.

Lemma 3 (Maximal proofs are normal forms (left decomposition)): If \( D \) is maximal and \( D \Rightarrow E \), then \( D \approx_{\text{loc}} E \).

Proof: The proof is essentially identical to the original one [CMS08].

In each of the preemptive rewrite rules, a focusing rule is brought closer to the root of the derivation, and the rest of the structure is preserved. For any subderivation \( D' \) of \( D \), there is a corresponding subderivation \( E' \) of \( E \) with \( D' \Rightarrow E' \) – even if no rewrite happened in this subderivation, we take \( E' = D' \). By soundness we have \( D' \approx_{\text{glob}} E' \) and, by maximality, \( \text{roots}(D') \supseteq \text{roots}(E') \). So we have \( \text{roots}(D') = \text{roots}(E') \): the multi-focusing rules in \( D \) and \( E \) are equivalent modulo local premutations. As the rest of the rules also are, we have \( D \approx_{\text{loc}} E \).

To our knowledge, the following lemma is not part of previous characterizations of maximal proofs.

Lemma 4 (Normal forms are maximal): If \( D \) is such that \( D \approx_{\text{loc}} E \) whenever \( D \Rightarrow E \), then \( D \) is maximal.

Proof: By contraposition, let us assume that \( D \) is not maximal and rewrite it into some \( E \) such that \( D \not\approx_{\text{loc}} E \).

If \( D \) is not maximal, there is a subderivation \( D' \) of it and a counter-example \( E' \) such that \( D' \approx_{\text{glob}} E' \) but \( \text{roots}(E') \not\subseteq \text{roots}(D') \). We call interesting the foci that are in \( \text{roots}(E') \) but not in \( \text{roots}(D') \) – there is at least one. Given that those two subproofs are globally equivalent, they can still be found somewhere in \( D' \), just not in the first multi-focusing rule; they are in another, separated by a non-empty sequence of invertible rules – as those are proofs in the non-preemptive system.

We claim that these interesting foci could be moved down in \( D' \) along with their complete sequences of non-invertible rules. A focus can only be moved down if it does not depend on the rules below, or if they can themselves be moved down. But we know that they do not depend on any other focus of \( D' \), as they are present in the very first multi-focusing rule of \( E' \) and global rule premutations do preserve dependencies. We can therefore move them to the root of \( D' \).

This gives a new sub-proof, \( D'' \), that is globally equivalent to \( E' \) and \( D' \), but not locally equivalent to \( D' \): we permuted non-invertible rules below invertible rules. We conclude by defining \( E \) as \( D[D''/D'] \), or \( D \) where its subproof \( D' \) is replaced by \( D'' \). We have \( D' \Rightarrow D'' \), hence \( D \Rightarrow E \), but \( D \not\approx_{\text{loc}} E \).

These two lemmas provide a computable characterization of maximal proofs: they are the normal forms of the preemptive rewriting relation.

Theorem 1 (Characterization of maximality): Maximal proofs are exactly the normal form, up to local equivalence, of the rewriting relation.

We can also prove that they are unique modulo global equivalence.

Lemma 5 (Completeness of preemptive rewriting (right decomposition)): If \( D \approx_{\text{glob}} E \) and \( E \) is maximal, then \( D \Rightarrow E \).

Proof: As in the original paper [CMS08]. We proceed by case analysis on rule premutations that send a focus down. For each,
we can verify it is of the preemptive rewriting rules. □

Theorem 2 (Canonicity): If two maximally multi-focused proofs are globally equivalent, they are in fact locally equivalent.

Proof: As in the original paper [CMS08]: if \( D \approx_{\text{glob}} E \) are both maximal, then \( D \Rightarrow E \) by completeness (Lemma 5), but the maximal proof \( D \) is a normal form (Lemma 3) so \( D \approx_{\text{loc}} E \). □

Corollary 1: Two multi-focused proofs are globally equivalent if and only if they are rewritten by \( \Rightarrow \) in locally equivalent maximal proofs.

III. ON THE SIDE OF PROOF TERMS
A. Preemption and reinversion as term rewriting

Now that we have a notion of maximally multi-focused proofs in natural deduction, we can cross the second bridge between multi-focusing and Lindley’s work by moving to a term system. We define below a term syntax for multi-focused derivations in natural deduction, and express preemptive rewriting and reinversion as term-to-term rewriting.

As the distinction between the preemptive and the non-preemptive systems are mostly about invariants of the focusing rule, the same term calculus is applicable to both. The only syntactic difference is the re-focusing rules inside elimination-focused rules in the preemptive system, which corresponds to the focusing case for negative terms.

Structural constraints on the multi-focusing system (preemptive or not) guarantee that strong typing invariants are verified. In particular, in a focused term \( \{ \bar{x} = \bar{n} \in p; t \} \), the \( \bar{n} \) must have a positive type, so the \( \bar{x} \) are always bound to positive types – only \( \lambda x. \) may introduce an hypothesis of negative type in the typing environment.

Here are the rewriting rules corresponding to the preemptive rewriting relation:

\[
\begin{align*}
t & ::= x, y, z & \text{variable} \\
& | \lambda x. t & \text{lambda} \\
& | (t, t) & \text{pair} \\
& | \delta(x, x, t, x, t) & \text{case} \\
& | f t & \text{focusing} \\
n & ::= x, y, z & \text{variable} \\
& | \pi_i n & \text{pair projection} \\
& | n t & \text{function application} \\
& | f n & \text{focusing (only in the preemptive calculus)} \\
f & ::= \text{multi-focusing} \\
& | \text{let } \bar{x} = \bar{n} \text{ in } p; & \text{multi-focus} \\
& | \text{let } \bar{x} = \bar{n} \text{ in } & \text{multi-focus (with no positive focus)} \\
p & ::= \ast & \text{identity} \\
& | \sigma_i p & \text{sum injection} \\
\end{align*}
\]

Fig. 5. Syntactic definitions

Lemma 6: If \( t \) is a proof term for the preemptive derivation \( D \), then \( t \rightarrow u \) if and only if \( u \) is a proof term for a preemptive derivation \( E \) with \( D \rightarrow E \).

Proof: Immediate, by inspection of the two rewriting system. □

The reinversion relation also has a corresponding term-rewriting interpretation. To perform each invertible rule as early as it should be, it suffices to let any reversible rule skip over a non-invertible phase it does not depend on. Depending on the order of the invertible rules after this phase, the invertible rule we want to move may be after a series of invertible rules that cannot be moved. We capture this by reducing invertible rules happening inside contexts of the form \( C_{ni}[C_i[ \_ ] \_ ] \), where \( C_i[t] \) is a notation for invertible contexts (defined using invertible frames \( F_i[t] \), and \( C_{ni}[t] \) for non-invertible contexts. Defining the latter requires describing negative/elimination contexts \( C_{neg}[t] \), with holes where a term may appear in a series of elimination-focused terms.

\[
\begin{align*}
F_i[ \_ ] & ::= \lambda x.[ \_ ] \\
& | \delta(x, x, x_1.[ \_ ], x_2.t) \\
& | \delta(x, x, x_1.t, x_2.[ \_ ]) \\
& | (t, [ \_ ]) \\
& | ([ \_ ], t) \\
C_{neg}[ \_ ] & ::= n \_ \\
& \_ t \\
& \_ C_{neg}[ \_ ] \\
& \_ C_{ni}[C_{neg}[ \_ ]] \\
C_i[ \_ ] & ::= [ \_ ] | F_i[C_i[ \_ ]] \\
C_{ni}[ \_ ] & ::= \text{let } \bar{x} = \bar{n} \text{ in } p; \_ [ \_ ] \\
& \_ \text{let } \bar{x}, y = \bar{n}, C_{neg}[ \_ ] \text{ in } p; \_ t
\end{align*}
\]

Reinversion can then be precisely defined as the transitive
The rewrite conditions are expressed in terms of a $C[·] \prec c$ relation (read "context $C$ blocks term-constructor $c$") that indicates a dependency of an invertible construction $c$ on a given context $C[·]$. For example, it would make no sense to extrude a $\lambda$ in argument position in a destructor, or move a sum-elimination $\delta(x)$ across the frame that defined the variable $x$.

$$\begin{align*}
\text{let } \bar{x} = \bar{n} \text{ in } p; & \quad \begin{cases}
y \varepsilon \bar{x} \\ y \notin \bar{x}
\end{cases} & \quad \delta(y) \\
\text{let } \bar{x} = \bar{n} \text{ in } p; & \quad \begin{cases}
y \notin \bar{x} \\
\neq \emptyset
\end{cases} & \quad (\bar{x}) \mid \lambda
\end{align*}$$

$$\begin{align*}
\text{let } \bar{x}, y = \bar{n}, C_{neg[·]} & \text{ in } p; \quad ?t C_{neg[\bar{n}]} \prec c \\
\langle [\bar{x}], t \rangle \mid \langle t, [\bar{x}] \rangle \mid \lambda & \xleftarrow{\prec} (\bar{x}) \mid \lambda \\
\lambda x. & \xleftarrow{\prec} \delta(x) \\
\delta(x, y. [\bar{x}], z. t) & \xleftarrow{\prec} \delta(y)
\end{align*}$$

$$\begin{align*}
\text{let } \bar{x}, y = \bar{n}, C_{neg[·]} & \text{ in } p; \quad ?t C_{neg[\bar{n}]} \prec c \\
\bar{n} \mid \pi_i & \xleftarrow{\prec} (\bar{x}) \mid \lambda \\
C_{neg[\bar{n}]} t \mid \pi_i C_{neg[\bar{n}]} C_{ni[\bar{n}]} & \xleftarrow{\prec} c \\
C_{ni[\bar{n}]} & \xleftarrow{\prec} c \\
F_i[C_{ni[\bar{n}]}, F_i[\bar{n}] & \xleftarrow{\prec} c \\
F_i[C_{ni[\bar{n}]}, C_{ni[\bar{n}]}, C_{ni[\bar{n]}}, C_{ni[\bar{n}]}, & \xleftarrow{\prec} c
\end{align*}$$

It may at first seem surprising that reinversion rules have instances that are the opposite of some of the preemptive rewriting rules — those about pos/neg permutations. But that is precisely one of the purposes of reinversion: after preemptive rewriting rules have been fully applied, we undo those that have gone "too far", in the sense that they let a non-invertible phase preempt a portion of an invertible phase below, but were blocked by dependencies without reaching the next non-invertible phase. This blocked phase does not increase the parallelism of multi-focusing in the proof, but stops the derivation from being valid in the original multi-focusing system, so reinversion undoes its preemption.

Remark, in relation to this situation, that preemptive rewriting cannot be easily defined on equivalence classes of neg/neg permutations (or other presentations of focusing that crush the invertible phase in one not-so-interesting step, such as higher-order focusing), as the order of the invertible rules in a single phase may determine where a non-invertible phase stops its preemption and is blocked in the middle of the invertible phase. Reinversion restores this independence on invertible ordering. This explains why the meta-theory of maximal multi-focusing was conducted in the non-preemptive system, using the relation between proofs that always applies reinversion after preemptive rewriting.

The other interesting case is a non-invertible phase $\pi_0$ having traversed a family of non-invertible phases $(\pi'_i)_{i \in I}$, before merging into some non-inversible phase $\pi_1$. Reinversion will move its negative phase $\nu_0$, reverting the preemption of the $\pi'_i$ on the invertible formulas introduced by $\pi_0$. But the important preemptions that happened, namely the traversal by $\pi_0$ of each of the invertible phases $(\nu'_i)_{i \in I}$, are not reverted: each $\nu'_i$ is blocked by the $\pi'_i$ below and thus cannot be reverted below $\pi_0$. As $\pi_0$ traversed both the $\nu'_i$ and the $\pi'_i$, it does not have the corresponding invertible formulas in its context anymore, and is well-positioned even in a non-preemptive proof.

**Lemma 7:** If $t$ is the proof term of the preemptive derivation $D : \Gamma \vdash A$, and $u$ is such that $t \triangleright u$, then $u$ is a valid (preemptive) proof term for $\Gamma \vdash A$.

**Proof:**

We will list here the derivation-level transformations that correspond to reinversion rewriting. We express them as a judgment $\Gamma \vdash A \triangleright F$, where $F$ is a invertible multi-hole context $((\lambda x_\cdot [\bar{x}], [\bar{x}])$ or $\delta(x, y. [\bar{x}], z. t)$) that is allowed to permute below a derivation for $\Gamma \vdash A$. The derivations for this judgment correspond to the (context, constructor) pairs that are absent from the blocking relation $\prec$.
\[
\begin{align*}
\Gamma \vdash \Delta \quad A \vdash A' & \quad \Gamma, \Delta \vdash A \triangleright c \quad c \in \{(), \lambda\} \\
\Gamma \vdash A \triangleright c \\
\Gamma, x : A + B \vdash y : \Delta \\
\Gamma \vdash x : A + B, \Delta \triangleright D \triangleright \delta(x) \\
\Gamma, x : A + B \vdash C \triangleright \delta(x)
\end{align*}
\]

\(\exists \bar{y} \in \bar{y}
\]

This shows preservation of preemptive well-typing as one then simply has to check that each inference rule

\[
\frac{\Gamma' \vdash A' \triangleright F}{\Gamma \vdash A \triangleright F}
\]

verifies that if applying the rule corresponding to the constructor \(F\) is a correct way to start a proof of \(\Gamma' \vdash A'\), then it is also a correct way to start a proof of \(\Gamma \vdash A\).

By a tedious case analysis, we can see that \(\triangleright\) preserves typing.

\(\square\)

**Lemma 8:** If \(u\) is a valid proof term in the preemptive system, and a normal form of the relation \(\triangleright\), then \(u\) is also a valid proof term for the non-preemptive system.

**Proof:** By contraposition, we show that if \(u\) does use preemption, that is if a multi-focusing happens while an invertible rule is applicable, then \(\triangleright\) can rewrite \(u\) into a different proof. A preemptive focusing is of the form

\[
\begin{align*}
\Gamma_n, \Sigma_p \vdash \Delta_p \\
A \vdash^2 B \\
\Gamma_n, \Sigma_p, \Delta \vdash B \\
\Gamma_n, \Sigma_p \vdash A
\end{align*}
\]

where either \(\Sigma\) is non-empty, or \(A\) is negative (and \(A = B\)). Supposing \(\Sigma\) is non-empty, there is a non-atomic positive formula in the context of our derivation. It cannot be used by the axiom rule, which requires the variable type to be atomic, nor in the variable rule of the elimination-focused judgment which only uses negative formulas. If it is used, it is in an invertible sum-elimination judgment higher in the derivation. This judgment either happens in the invertible phase directly above the current focusing, or later, after some other focusing.

higher in the derivation. In any case, the positive formula is in the context of this last focusing, and can be inverted just below it, which produces a distinct inverted proof.

(If the positive hypothesis is never used, we could claim that weakening the derivation to remove it gets us closer to a valid preemptive proof. Alternatively, we have a negativity restriction on the context of the axiom rule, which enforces that it must be eliminated somewhere in any complete proof. We prefer to avoid discussing this axiom rules and potential atomic polarity assignment, as those concerns are orthogonal to the present work; we insist that this “negative axiom context” technique is not crucial to the present proof.)

Similarly, if the succeedent of the focusing rule is a non-atomic negative formula, then it is either introduced in the following invertible phase, or one just after some focusing rule of the same succeedent – in any case, it must be introduced before the axiom rules which requires an atomic succeedent. We can move that inversion rule below the closest focusing, which produces a distinct inverted proof.

\(\square\)

This gives us a precise characterization on proof terms of the relation \(\Rightarrow\).

**Theorem 3:** If \(t\) is a proof term for \(D\) and \(u\) for \(E\), then \(D \Rightarrow E\) if and only there is a \(u'\) such that \(t \rightarrow^* u' \triangleright u\), and \(u\) is a normal form for \(\triangleright\).

**B. Multi-focused terms as lambda-terms**

There is a natural embedding \([t]\) of a multi-focused term \(t\) into the standard lambda-calculus, generated by the following transformation, where \(t[x := \bar{u}]\) represents simultaneous substitution:

\[
\begin{align*}
[\text{let } x = \bar{n} \text{ in } p;^2 t] & := [p]^2([t][x := [\bar{n}]]) \\
[\emptyset](t) & := t \\
[\star](t) & := t \\
[\sigma_i p](t) & := \sigma_i[p](t)
\end{align*}
\]

The substitutions break the invariant that the scrutinee of a sum-elimination construct is always a variable. However, as only negative terms are substituted, sum-elimination scrutinees are always neutrals – embedding of negative terms. In particular, this embedding does not create any \(\beta\)-redex. Proof terms coming from non-preemptive multi-focusing are also always in \(\eta\)-long form, and this is preserved by the embedding.

**Fact 1:** If \(\Gamma \vdash t : A\) in the preemptive multi-focused system, then \(\Gamma \vdash [t] : A\) in simply-typed lambda-calculus, and \([t]\) is in \(\beta\)-normal form. If \(t\) is valid in the non-preemptive system, then \([t]\) is also in \(\eta\)-long form.

**C. Lindley’s rewriting relation**

The strong \(\eta\)-equivalence for sums makes lambda-term equivalence a difficult notion. For any term \(m : A + B\) and well-typed context \(C[\ ]\), it dictates that \(C[m] \simeq \delta(m, x_1, C[x_1], x_2, C[x_2])\). In his article [Lin07], Sam Lindley breaks it down in four simpler equations, including in particular the “weak”, non-local \(\eta\)-rule (where \(F\) represent a frame,
that is a context of term-size exactly 1):
\[ m \approx \delta(m, x_1, \sigma_1, x_1, x_2, 0_2, x_2) \quad (+, \eta) \]
\[ F[\delta(p, x_1.t_1, x_2.t_2)] \approx \delta(p, x_1.F[t_1], x_2.F[t_2]) \quad \text{(move-case)} \]
\[ \delta(p, x_1, \delta(p, y_1.t_1, y_2.t_2), x_2, \delta(p, z_1, u_1, z_2, u_2)) \approx \text{ (repeated-guard)} \]
\[ \delta(p, x_1.t_1[y_1 := x_1], x_2, u_2[2 := x_2]) \]
\[ \delta(p, x_1.t_1, x_2.t_2, x_1, x_2 \not\approx t) \quad \text{(redundant-guard)} \]

Lindley further refines the move-case equivalence into a less-local hoist-case rule. Writing \( D \) for a frame that is either \( \delta(p, x_1.t_1, x_2.t_2) \) or \( \delta(p, x_1.t_1, x_2.t_2) \), \( D* \) for an arbitrary (possibly empty) sequence of them, and \( H \) any frame that is not of this form, hoist-case is defined as:
\[ H[D^*[\delta(t, x_1.t_1, x_2.t_2)] \to \delta(t, x_1.H[D^*[t_1]], x_2.H[D^*[t_2]]) \]

Lindley’s equivalence algorithm (Theorem 36, p. 13) proceeds in three steps: rewriting terms in \( \beta, \eta, \gamma \)-normal forms (using the weak \( (+, \eta) \)) on sums, then rewriting them in \( \gamma \)-normal form, and finally using a decidable redundancy-eliminating equivalence relation called \( \sim \). The rewriting relation \( \gamma \) is defined as the closure of repeated-guard, redundant-guard (when read left-to-right) and hoist-case; \( \gamma \) is a weak restriction of it defined below. The equivalence \( \sim \) is the equivalence closure of the equivalence repeated-guard, redundant-guard, and move-case restricted to \( D \)-frames – clauses of a sum elimination.

We discuss redundancy elimination, that is aspects related to repeated-guard and redundant-guard, in Section [IV] and focus here on explanation of the other rewriting processes (\( \beta, \eta, \gamma \) and hoist-case) in logical terms. We show that multi-focused terms in \( \Rightarrow \)-normal form embed into \( \beta, \eta, \gamma \)-\( \gamma \)-normal forms.

As we ignore the redundancy-elimination rules, our result is only established modulo \( \sim \).

The \( \beta \) and \( \eta \) rewriting rules are standard – for sums, this is the weak, local \( \eta \)-relation, and not the strong \( \eta \)-equivalence. As explained in the previous subsection, embeddings of proof terms valid in the non-preemptive system – as are \( \Rightarrow \)-normal forms – are in \( \beta, \eta \)-normal form.

The rewriting \( \gamma \) is defined as the extrusion of a sum-elimination out of an elimination context: \[ [ ] t \mid \pi_1[ ] \mid \delta([], x_1.t, x_2.t) \] It is, again, subsumed by the static structure of valid focused proofs: contexts \([ ] t \) and \( \pi_1[ ] \) may only contain negative terms, which exclude sum eliminations, and variables of sum-elimination scrutinees may only be bound to negative terms, so their embedding cannot be a sum elimination either.

Fact 2: Terms for valid preemptive multi-focusing derivations are in \( \gamma \)-normal form.

This rigid structure of focused proofs is well-known, just as \( \beta, \eta \)-normality or commuting conversions are not the interesting points of Lindley’s work. The crux of the correspondence is between the transformation to maximal proofs, computed by \( \Rightarrow \), and his \( \gamma \)-rewriting relation. There is an interesting dichotomy:

- Preemptive rewriting, which merges non-invertible phases, is where most of the work happens from a logical point of view. Yet this transformation, on the embeddings of the multi-focused proof terms, corresponds to the identity!
- Reversion, which is obvious logically as it only concerns invertible rules which commute easily, corresponds to \( \gamma \)-rewriting on the embeddings.

Of course, preemptive rewriting is in fact crucial for \( \gamma \)-rewriting. It is the one that determines up to where negative terms can move in the derivation, and in particular the scrutinees of sum eliminations. Reversion would not work without the first preemptive rewriting step, and applying reversion on a proof term that is not in preemptive-normal form may not give a \( \gamma \)-normal embedding.

**Lemma 9:** If \( t \to u \), then \( [t] = [u] \).

**Proof:** Immediate by inspection of the term-level preemptive-rewriting rules.

**Lemma 10:** If \( t \to u \), then \( [t] \to^* [u] \).

**Proof:** Our reversion contexts \( C_n[C_i[ ]] \) were naturally motivated by the translation from one logical system to another – from preemptive to non-preemptive proofs; yet they closely correspond to Lindley’s notion of hoisting contexts \( H[D^*[ ]] \). However, while we permute all the syntactic construction corresponding to invertible rules, in particular lambdas and pairs, the hoist-case rule only moves sum-eliminations. We can however show that our other permutations are invisible on the embedded term:

\[
C_n[C_i[\lambda x.t]] \to^* \lambda x. C_i[C_i[t]]
\]

This crucially relies on the blocking relation: we can immediately verify that for any \( C_n[C_i[ ]] \) such that \( [C_n[C_i[\lambda x.t]]] \neq [\lambda x. C_n[C_i[t]]] \), we have \( C_n[C_i[ ]] \sim \lambda \), and therefore the \( \beta \)-normalization hypothesis cannot hold – and similarly for pairs.

In the last case where we hoist sum-eliminations, we still have to be careful because of the non-locality of the embedding \([t] \): negative terms at the very root of the focused proofs may find themselves substituted very far away in the lambda-term. The interesting case is the following rewrite, where \( E[ ] \) is an arbitrary term context with \( x \notin E \):

\[
\text{let } x = \delta(y, y_1.t_1, y_2.t_2) \quad \text{in } E[x]
\]

While this rewrite is very local in nature, the corresponding embedding is not:

\[
E[\delta(y, y_1.t_1, y_2.t_2)] \to^* \delta(y, y_1.E[t_1], y_2.E[t_2])
\]

The sum-elimination has been extruded out of the whole context \( E \), which may be arbitrarily large and use arbitrary term constructors. This is allowed by \( \to^* \). Furthermore, if \( E \) is not linear (the hole \([ ] \) occurs several times in \( E \)), what is a single sum-elimination extrusion in
the multi-focused term may in fact translate into several sum-eliminations (on the same scrutinee) in its embedding. Just applying hoist-case as many times does not produce the desired reduced term, we also have to use repeated-guard. □

Lemma 11: If \( u \) is in \( \Rightarrow \)-normal form, then for some \( u' \approx_{\text{loc}} u \), \( |u'| \) is in \( \gamma \)-normal form modulo \( \sim \).

Proof: Even if we have \( |u| \Rightarrow \gamma m \) for some \( m \), it is impossible that a case elimination of \( |u| \) could be hoisted over the constructor of a non-invertible rule: that would amount to permuting an invertible rule below a non-invertible phase, which would allow to make a step of \( \triangleright \)-reduction from \( u \) — assuming that the preemptive rewrites have been fully applied, which is the case here. Any step of reduction from \( |u| \) then corresponds to permuting a sum-elimination over a lambda or a pair constructor inside an invertible phase. We can apply those simplifications fully, and get a \( u' \approx_{\text{loc}} u \) such that \( |u'| \) is in \( \gamma \)-normal form. □

Theorem 4 (\( \gamma \)-normal forms are embeddings of maximally-focused proofs): If \( \{ t \} \Rightarrow \gamma n \) and \( n \) is \( \gamma \)-normal, then there are \( u \approx_{\text{loc}} u' \) such that \( t \Rightarrow u \) and \( |u'| \sim n \). In particular, \( u \) is maximally multi-focused.

Proof: Simply using \( \triangleright \) on \( t \) is not enough, as a sum-elimination may then be blocked by where the focus on its scrutinee happens in \( t \), while \( \Rightarrow \gamma \) would still be able to move the elimination and its scrutinee below in the proof. We have to first rewrite \( t \) into \( t' \), a normal form for preemptive rewriting, and then apply reversion completely on \( t' \) to get \( u \). We have to rewrite \( t = |t'| \) from Lemma 2 and \( t' \Rightarrow |u| \) from Lemma 10. Applying Lemma 11 gives us \( u' \approx_{\text{loc}} u \) such that \( |u'| \) is a \( \gamma \)-normal form modulo \( \sim \). The term \( n \) is also \( \gamma \)-normal, and both can be reached from \( |t| \). Lindley proved confluence of \( \gamma \)-rewriting modulo \( \sim \) (Proposition 28, p. 11), and we therefore have \( |u| \sim n \) as desired. □

IV. REDUNDANCY ELIMINATION

In the previous section, we have glossed over the fact that Lindley’s \( \gamma \)-reduction also simplifies redundant and duplicated sum-eliminations. This is not part of the operations on multi-focused proofs we have defined so far, which tried to follow previous multi-focused systems as closely as possible. In this section, we will enrich our multi-focused system to simplify away redundant foci and duplicated sum-elimination, to make a system that is completely equivalent to Lindley’s.

We simply have to add the following simplifications on proof terms:

REdundant-focus

\[
\begin{align*}
&\text{REDUNDANT-FOCUS} \\
&\text{let } \overline{x}, y, z = \overline{n}, n', n'' \text{ in } p; \overline{s} t \Rightarrow s \text{ let } \overline{x}, y = \overline{n}, n' \text{ in } p; \overline{s} t[z := y] \\
\end{align*}
\]

REdundant-guard

\[
\begin{align*}
&\text{REdundant-guard} \\
&\delta(x, x_1.t, x_2.t) \approx_{\text{loc}}^{|s|} t \\
\end{align*}
\]

Repeated-case-1

\[
\begin{align*}
&\text{REPEATED-CASE-1} \\
&\delta(x, x_1.\delta(x, y_1.u_1, y_2.u_2), x_2.t_2) \approx_{\text{loc}} \delta(x, x_1.u_1[y_1 := x_1], x_2.t_2) \\
\end{align*}
\]

Repeated-case-2

\[
\begin{align*}
&\text{REPEATED-CASE-2} \\
&\delta(x, x_1.t_1, x_2.\delta(x, y_1.u_1, y_2.u_2)) \approx_{\text{loc}} \delta(x, x_1.y_1, x_2.u_2[y_2 := x_2]) \\
\end{align*}
\]

Those rules are reasonable in a focusing setting, as they respect the phase separation. While rules that test for equality of subderivations have an unpleasant non-local aspect, they are common in the literature on sum equivalence (Lindley [Lin07], or Balat, Di Cosmo and Fiore [BCF04], have a similar test in their normal form judgments), and has also been used previously in the multi-focusing literatures, for other purposes; in Alexis Saurin’s PhD thesis [Sau08], it is used to give a convenient \( \otimes/\& \) permutation rule (p. 231).

One should however remark that they break the property of preserving the initial sequents of proofs (when seen as a multiset), property which was carefully preserved by our previous notions of equivalence. However, as we can apply these simplifications after reduction to maximally multi-focused proof, they do not interfere with previous canonicity results for maximal multi-focusing.

Definition 6: We define the relation \( t \Rightarrow_{s} u \) between proof terms of the (preemptive) multi-focused calculus as follows:

\[
\begin{align*}
&t \Rightarrow_{s} t_1 \Rightarrow_{s} t_2 \Rightarrow_{s} t_0 \approx_{\text{loc}} u \\
\end{align*}
\]

where \( t_1 \) is a preemptive normal form, \( t_2 \) is a redundant-foci normal form, and \( u_0 \) is a \( \triangleright \)-normal form.

Definition 7: We call the \( u \) that are the target of the \( \Rightarrow_{s} \) relation simplified maximal forms.

Having embedded redundancy-elimination in the definition of maximal forms, we can now get strong correspondence results between \( \Rightarrow_{s} \)-normal and \( \gamma \)-normal forms, without needing a modulo-\( \sim \) provision.

Theorem 5 (Simplified maximal forms are \( \gamma \)-normal): Given a multi-focused term \( t \), there exists some \( u \) such that \( t \Rightarrow_{s} u \), \( |t| \Rightarrow \gamma |u| \), and \( |u| \) is in \( \gamma \)-normal form. This \( u \) is unique modulo local equivalence.

Proof: We have already shown that \( \Rightarrow_{s} \)-normal forms are \( \gamma \)-normal form modulo redundancy elimination and hoisting of \( \delta \) over \( \lambda \). A \( \Rightarrow_{s} \)-normal form is a \( \Rightarrow_{s} \)-normal form where repeated-guard and redundant-guard have been eliminated. Furthermore, in the \( \approx_{\text{loc}} \)-equivalence class of proofs \( u \) such that \( t \Rightarrow_{s} u \), we can pick the one in which invertible phases always have all sum-eliminations before any lambda or pair constructor. By construction, \(|u| \) is then a \( \gamma \)-normal form. Note that it is not completely obvious, a priori, that \( \Rightarrow_{s} \)-normal forms always embed into \( \gamma \)-normal form, and this crucially relies on the REDUNDANT-FOCUS rule. Indeed, consider for example the term \( y_1, y_2 = f x, f x \in \delta(z, z_1.y_1, z_2.y_2) \). While it is in REDUNDANT-GENERAL-normal form, its embedding \( \delta(z, z_1.f x, z_2.f x) \) is not. The source of the problem is that redundancy rules rely on term equality, and we may have \( t \neq u \) but \(|t| = |u| \).

We can prove however that maximal proofs that are REDUNDANT-FOCUS-normal forms are normal for the REPEATED-GUARD or REDUNDANT-GUARD only if their embedding is as well. It suffices to prove that if \( t, u \) have no redundant cases, then \(|t| = |u| \) implies \( t = u \). Consider for example terms of the form \( C[\pi_1, x], C[\pi_2, y] \), where \( C \) is a common context — the following proof generalizes to all other cases. If \( C[\pi_1, x] = C[\pi_2, y] \), there is a substitution \( \sigma \)
such that \( x[\sigma] = y[\sigma] \), which means that either \( x = y \) or they have been substituted terms of equal embedding: \( \sigma \) is of the form \( \{ x := [n_1], y := [n_2], \ldots \} \) with \( [n_1] = [n_2] \). Inductively (the chain of nested substitutions is finite), we can assume that \( n_1 = n_2 \). As our terms are maximally preempted, the definitions let \( x = n_1 \) and let \( y = n_2 \) in therefore happen in the same non-invertible phase (\( n_1 \) and \( n_2 \) being equal, they have the same dependencies), and can be merged by the REDUNDANT-FOCUS rule.

**Corollary 2:** Two multi-focused proof terms are extensionally equivalent if their maximally multi-focused normal forms are locally equivalent (modulo redundancy elimination).

**RELATED AND FUTURE WORK**

Maximally multi-focused proofs were previously used to bridge the gap between sequent calculus, as a rather versatile way of defining proof systems, and ad-hoc proof structures designed to minimize redundancy for a fixed logic. The original paper on multi-focusing [CMS08] demonstrated an isomorphism between maximal proofs and proof nets for a subset of linear logic. In recent work [CHM12], maximally multi-focused proof of a sequent calculus for first-order logic have been shown isomorphic to expansion proofs, a compact description of first-order classical proofs.

There are some recognized design choices in the land of equivalence-checking presentation that can now be linked to design choices of focused system. For example, Altenkirch et al. [ADHS01] proposed to make the syntax more canonical with respect to redundancy-elimination by using a n-ary sum elimination construct, while Lindley prefers to quotient over local reorderings of unary sum-eliminations. This sounds strongly similar to the choice between higher-order focusing (\cite{Zeil09}), where all invertible rules are applied in a single step, or the quotienting of concrete proofs by neg/neg permutations as used here.

The restriction of sum-elimination in our focused proof term to only have variable as scrutinees may be reminding of term-presentation for sequent calculus rather than natural deduction. We consider instead that it is a direct result of the focusing discipline – foci are naturally represented by variables. Ironically, this brings us rather close to the sequent calculus of Krishnaswami \cite{Kri09} which, for presentation purposes, preserved a function-elimination rule in natural deduction style.

For the purpose of this work, but absent for lack of space, we investigated the proof-term presentation of preemptive rewriting on proof terms for sequent calculi, instead of natural deduction, in the spirit of Curien-Herbelin System L. The results are encouraging and seem to indicate that sequent-terms may be a natural medium for multi-focusing – for example, the blocking relation of reinversion would be easier to setup. In the future, we would like to investigate relations between maximally multi-focused proofs and techniques relying on extensional equality on the programming end of the Curry-Howard spectrum.

**CONCLUSION**

We propose a multi-focused calculus for intuitionistic logic in natural deduction, and establish the canonicity of maximally multi-focused proofs by transposive the preemptive rewriting technique [CMS08] in our intuitionistic, natural deduction setting. By studying the computational effect of preemptive rewriting on proof terms, we demonstrate the close correspondence with the rewriting on lambda-terms with sums proposed by Lindley [Lin07] to compute extensional equivalence. Adding a notion of redundancy elimination to our multi-focused system makes preemptive rewriting precisely equivalent to Lindley’s \( \gamma \)-rules. In particular, the resulting notion of canonical forms, simplified maximal proofs, captures extensional equality.

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**REFERENCES**


