# Multi-focusing on extensional rewriting with sums (introduction)

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- Sum equivalence looks hard. Can we implement it?
- Are there representations of programs (proofs) that quotient over those equivalences?

# My paper in one slide

The equivalence algorithm of

Sam Lindley.
Extensional rewriting with sums.
In *TLCA*, pages 255–271, 2007.

and the normalization of proof representations in

Kaustuv Chaudhuri, Dale Miller, and Alexis Saurin.
Canonical sequent proofs via multi-focusing.
In *IFIP TCS*, pages 383–396, 2008.

are doing (almost) the same thing
– and we had not noticed.

# In this talk

Sam Lindley's rewriting-based algorithm is the first **simple** solution (first solution: Neil Ghani, 1995) to deciding sum equivalences.

It's easy to understand and follow. But to me it felt a bit arbitrary.

On the other hand, (multi-)focusing is beautiful, but requires some background knowledge.

Providing it is the purpose of this talk.

# Sequent calculus

(Can be done in natural deduction, but less regular)

$$\frac{\Gamma \vdash A \qquad \Gamma, B \vdash C}{\Gamma, A \to B \vdash C} -$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}$$

$$\frac{\Gamma, A_i \vdash C}{\Gamma, A_1 * A_2 \vdash C} -$$

$$\frac{\Gamma \vdash A_1 \qquad \Gamma \vdash A_2}{\Gamma \vdash A_1 * A_2}$$

$$\frac{\Gamma, A_1 \vdash C \qquad \Gamma, A_2 \vdash C}{\Gamma, A_1 + A_2 \vdash C}$$

$$\frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 + A_2} +$$

Inversible vs. non-inversible rules.
Negatives (interesting on the left): products, arrow, atoms.
Positives (interesting on the right): sum, atoms.

# Inversible phase

$$\dfrac{\dfrac{?}{X + Y \vdash X}}{X + Y \vdash X + Y}$$

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**Focusing restriction 1: inversible phases**

Inversible rules must be applied as soon and as long as possible
– and their order does not matter.

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**Focusing restriction 1: inversible phases**

Inversible rules must be applied as soon and as long as possible
– and their order does not matter.

Imposing this restriction gives a single proof of $(X \to Y) \to (X \to Y)$ instead of two ($\lambda(f)\, f$ and $\lambda(f)\, \lambda(x)\, f\ x$).

# Non-inversible phases

After all inversible rules, $\Gamma_n \vdash A_p$

Only step forward: select a formula, apply some non-inversible rules on it.

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**Focusing restriction 2: non-inversible phase**

When a principal formula is selected for non-inversible rule, they should be applied as long as possible – until its polarity changes.

Completeness: this restriction preserves provability. **Non-trivial !**
Example of removed redundancy:

$$\dfrac{\dfrac{\dfrac{X_2, \quad Y_1 \vdash A}{X_2 * X_3, \quad Y_1 \vdash A}}{X_2 * X_3, \quad Y_1 * Y_2 \vdash A}}{X_1 * X_2 * X_3, Y_1 * Y_2 \vdash A}$$

# This was focusing

Focused proofs are structured in alternating phases,
inversible (boring) and non-inversible (focus).

Phases are forced to be as long as possible – to eliminate duplicate proofs.

The idea is independent from the proof system.
Applies to sequent calculus or natural deduction;
intuitionistic, classical, linear, you-name-it logic.

On proof terms, these restrictions correspond to $\beta\eta$-normal forms (at least for products and arrows). But the fun is in the search.

# Demo Time

$$(1 \to X \to (Y + Z)) \to X \to (Y \to W) \to (Z + W)$$

# Restrictive syntax

So far we've defined **focused proofs** as a subset of proofs in our system. Some people prefer to give them a syntax that enforces their structure.

$$\frac{\Gamma; \Delta, A \vdash B}{\Gamma; \Delta \vdash A \to B} \qquad \frac{\Gamma; \Delta \vdash A \qquad \Gamma; \Delta \vdash B}{\Gamma; \Delta \vdash A * B}$$

$$\frac{\Gamma; A, \Delta \vdash C \qquad \Gamma; B, \Delta \vdash C}{\Gamma; A + B, \Delta \vdash C} \qquad \frac{\Gamma, A_n; \Delta \vdash C}{\Gamma; A_n, \Delta \vdash C}$$

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$$\frac{\Gamma_n \vdash \boxed{C_p}}{\Gamma_n; \emptyset \vdash C_p} \qquad \frac{\Gamma \| \boxed{A_i} \vdash C}{\Gamma \| \boxed{A_1 * A_2} \vdash C} \qquad \frac{\Gamma \vdash A \qquad \Gamma \| \boxed{B} \vdash C}{\Gamma \| \boxed{A \to B} \vdash C}$$

$$\frac{\Gamma \vdash \boxed{A_i}}{\Gamma \vdash \boxed{A_1 + A_2}}$$

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$$\frac{\Gamma \vdash \boxed{A_i}}{\Gamma \vdash \boxed{A_1 + A_2}} \qquad \frac{\Gamma; B \vdash C}{\Gamma \| \boxed{B_p} \vdash C} \qquad \frac{\Gamma; \emptyset \vdash C}{\Gamma \vdash \boxed{C_n}}$$

# Success stories

Focusing was introduced by Andreoli in 1992.
Revolution in logic programming.

Forward-chaining and backward-chaining expressed in a single system by assigning polarities to atoms.
Funnier stuff (magic sets?) with dynamic polarity changes.

Syntethic connectives: state-of-the-art automated theorem proving for non-classical logics
(+ Jumbo connectives, Paul Blain Levy, 2006)

Lazy vs. strict: focusing, polarization (Zeilberger (2008), Munch-Maccagnoni (2013)).

A sequent calculus with cut-free search bisimilar to DPLL (Lengrand, 2013).

# This is **not** the end

$$(A + A) \to A$$

$$(1 \to (A + A)) \to A$$

$$\lambda(x)\,\delta(x\,1,\ y.y,\ y.y)$$

$$\lambda(x)\,\delta(x\,1,\ y.\delta(x\,1,\ y'.y',\ y'.y'),\ y.y)$$

$$\lambda(x)\,\delta(x\,1,\ y.y,\ y.\delta(x\,1,\ y'.y,\ y'.y))$$

$$\dots$$

# Multi-focusing

Sometimes several independent foci are possible to make progress in a proof.

Multi-focusing (Miller and Saurin, 2007): do them all at once, in parallel.

$$\dfrac{\dfrac{\dfrac{X_2, \qquad Y_1 \vdash A}{X_2 * X_3, \qquad Y_1 \vdash A}}{X_2 * X_3, \quad Y_1 * Y_2 \vdash A}}{X_1 * X_2 * X_3, Y_1 * Y_2 \vdash A} \qquad \Rightarrow \qquad \dfrac{\dfrac{\dfrac{X_2, \qquad Y_1 \vdash A}{X_2 * X_3, \qquad Y_1 \vdash A}}{X_2 * X_3, \quad Y_1 * Y_2 \vdash A}}{X_1 * X_2 * X_3, Y_1 * Y_2 \vdash A}$$

$$\dfrac{\Gamma_n, \Delta_n \| \boxed{\Delta_n} \vdash A_p^?, \boxed{B_p^?}}{\Gamma_n, \Delta_n \vdash A_p^?, B_p^?}$$

# Maximal multi-focusing

Given a focused proof, it is possible to put focused sequences in parallel to exhibit some parallelism – without changing the operational meaning of the proof, seen as a pure program.

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Maximally multi-focusing is a powerful notion of canonical structure for proof.

- linear logic: proof nets (Chaudhuri, Miller, Saurin, 2008)
- first-order classical logic: expansion proofs (Chaudhuri, Hetzl, Miller, 2013)

"Evolution rather than revolution" (Dale Miller)

# Computing a maximal proof

**Preemptive** rewriting temporarily breaks the focused structure to move foci as far down as possible.

$$\left\{ \begin{array}{cc} & \mathrm{I}_3 \\ & \mathrm{NI}_3 \\ \mathrm{I}_2 & \\ \mathrm{NI}_2 & \\ \mathrm{I}_1 & \\ \mathrm{NI}_1 & \end{array} \right\}$$

$$? \left\{ \begin{array}{cc} \mathrm{I}_2 & \\ \mathrm{NI}_2 & \\ \mathrm{I}_1 & \mathrm{I}_3 \\ \mathrm{NI}_1 & \mathrm{NI}_3 \end{array} \right\}$$

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(This is the heart of the correspondence with Sam Lindley's work)

# Conclusion

Focusing imposes extra structure on proofs, based on rules permutability (inversible, non inversible).

Multi-focusing is a natural generalization of focusing, which gives very strong canonicity.

Existing equivalence-checking algorithms can be **logically justified** as maximalization techniques.

Multi-focusing may help me decide unicity of types.