The free bifibration over a functor

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Disclaimer

I'm the *least* category-competent of the authors of this work.

Don't expect good answers to your categorical questions. Sorry!

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One category living over another category, such that objects of the category above may be *pushed* and *pulled* along arrows of the category below.

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...and these liftings should be "universal" in an appropriate sense...

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$$
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S \arrow[r, "f_S"] & f^+ S \arrow[r, "f \backslash_g \alpha"] & T \\
A \arrow[r, "f"] & B \arrow[r, "g"] & C
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=
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$$
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S \xrightarrow{f_S} f^+ S \xrightarrow{f \backslash_g \alpha} T \\[2em]
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\end{array}
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\end{array}
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\begin{array}{c}
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\end{array}
$$

A few examples (from logic and computer science)

1. The forgetful functor $\mathcal{S}\mathrm{ubset} \to \mathcal{S}\mathrm{et}$ is a bifibration, where:

$$f^{+}(S \subseteq A) = f(S) \qquad f^{-}(T \subseteq B) = f^{-1}(T)$$

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   $\mathcal{D}$: objects are state predicates $P, Q \in \mathcal{P}(S)$, and morphisms in $P \to Q$ are given by programs $c : S \times S$ such that $\{P\}\ c\ \{Q\}$: $\forall (x, y) \in c, \quad x \in P \implies y \in Q$

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The forgetful functor from $\mathcal{D}$ to $\mathcal{C}$ has a bifibration
- $c^+\ P$ is the *strongest postcondition* of c from P: $\{y \mid \exists x \in P,\ (x, y) \in c\}$
- $c^-\ Q$ is the *weakest precondition* of c from Q: $\{x \mid \forall y \in Q,\ (x, y) \in c\}$

Problem

Given a functor, can we turn it into a bifibration in a universal way?

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This question has been relatively little-studied:

- R. Dawson, R. Paré, and D. Pronk (DPP). Adjoining adjoints. *Adv. Mathematics*, 2003.
- François Lamarche. Path functors in Cat. Unpublished, 2010. HAL-00831430.

Free bifibration: intuition

Introduce "formal" push/pull along the arrows of $\mathcal{C}$.

$$\mathsf{X} \xrightarrow{\delta \in \mathcal{D}} \mathsf{Y}$$

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& & {}
\end{tikzcd}$$

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Some operations commute: non-trivial equivalence.

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Under a condition coming from (DPP), *maximal* multi-focusing gives normal forms.

Finally, we found a couple *nice examples* of free bifibrations of a combinatorial nature.

A sequent calculus for $\mathcal{B}\mathrm{if}(p)$

Formulas / objects

Bifibrational formulas $S \sqsubset A$: $\quad S$ lies over A

$$\frac{X \in \mathcal{D} \quad p(X) = A}{X^{\eta} \sqsubset A} \qquad \frac{S \sqsubset A \quad f : A \to B}{f^{+}\, S \sqsubset B} \qquad \frac{f : A \to B \quad T \sqsubset B}{f^{-}\, T \sqsubset A} \qquad f^{+}\, g^{-}\, h^{-}\, Y$$

As diagrams: zigzags

$$(X \in D) \qquad \begin{tikzcd} S \arrow[d, "f"] \\ \cdot \end{tikzcd} \qquad \begin{tikzcd} T \\ \cdot \arrow[u, "f"] \end{tikzcd} \qquad \begin{tikzcd} Y \\ \cdot \arrow[u, "h"] \\ \cdot \arrow[u, "g"] \arrow[d, "f"] \\ \cdot \end{tikzcd}$$

Derivations / pre-morphisms

Axioms + inference rules $\alpha : S \underset{h}{\Longrightarrow} T$: $\quad \alpha : S \to T$ lies over $h : p(S) \to p(T)$.

$$\frac{\delta : X \to Y \in \mathcal{D} \qquad p(\delta) = f}{\delta^\eta : X^\eta \underset{f}{\Longrightarrow} Y^\eta}\ \delta^\eta$$

$$\frac{\alpha : T \underset{g}{\Longrightarrow} T'}{\overline{f}.\alpha : f^- \ T \underset{fg}{\Longrightarrow} T'}\ \mathsf{L}g^- \qquad\qquad \frac{\alpha : S' \underset{e}{\Longrightarrow} S}{\alpha.f : S' \underset{ef}{\Longrightarrow} f^+ \ S}\ \mathsf{R}f^+$$

$$\frac{\alpha : S \underset{fg}{\Longrightarrow} T}{f\backslash_g \ \alpha : f^+ \ S \underset{g}{\Longrightarrow} T}\ \mathsf{L}f^+ \qquad\qquad \frac{\alpha : S \underset{fg}{\Longrightarrow} T}{\alpha \ {}_f/\ \overline{g} : S \underset{f}{\Longrightarrow} g^- \ T}\ \mathsf{R}g^-$$

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$$\frac{\delta : X \to Y \in \mathcal{D} \qquad p(\delta) = f}{\delta^\eta : X^\eta \xRightarrow[f]{} Y^\eta}\ \delta^\eta \qquad \cdot \xrightarrow[f]{\delta} \cdot$$

$$\frac{\alpha : T \xRightarrow[g]{} T'}{\overline{f}.\alpha : f^- \, T \xRightarrow[fg]{} T'}\ \mathsf{L}g^-$$

$$\begin{tikzcd} \cdot \arrow[r, "g"'] & \cdot \arrow[d, equal] \\ \cdot \arrow[u, "f"'] \arrow[r, "fg"'] & \cdot \end{tikzcd}$$

$$\frac{\alpha : S' \xRightarrow[e]{} S}{\alpha.f : S' \xRightarrow[ef]{} f^+ \, S}\ \mathsf{R}f^+$$

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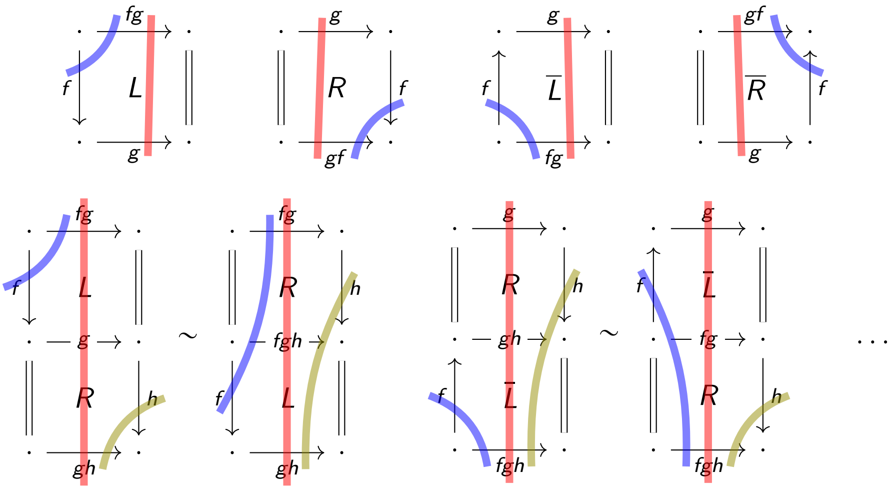
$$\begin{tikzcd} \cdot \arrow[r, "fg"'] \arrow[d, equal] & \cdot \\ \cdot \arrow[r, "f"] & \cdot \arrow[u, "g"] \end{tikzcd}$$

Permutation equivalences

$$
\begin{aligned}
(\overline{f}.\alpha).h &\sim \overline{f}.(\alpha.h) && \text{for } \alpha \text{ over } g && (1)\\
(f \backslash_g \alpha).h &\sim f \backslash_{gh} (\alpha.h) && \text{for } \alpha \text{ over } fg && (2)\\
(\overline{f}.\alpha)\ {}_{fg}/\ \overline{h} &\sim \overline{f}.(\alpha\ {}_{g}/\ \overline{h}) && \text{for } \alpha \text{ over } gh && (3)\\
(f \backslash_{gh} \alpha)\ {}_{g}/\ \overline{h} &\sim f \backslash_g (\alpha\ {}_{fg}/\ \overline{h}) && \text{for } \alpha \text{ over } fgh && (4)
\end{aligned}
$$

(reminiscent of Lambek calculus)

String diagrams



Identity and Composition

Identity (induction on the formula):

$$\mathsf{id}_{X^\eta} \stackrel{\text{def}}{=} (\mathsf{id}_X)^\eta \qquad \mathsf{id}_{f^+ S} \stackrel{\text{def}}{=} f \backslash_{\mathsf{id}_B} (\mathsf{id}_S . f) \qquad \mathsf{id}_{g^- T} \stackrel{\text{def}}{=} (\overline{g} . \mathsf{id}_T)\ {}_{\mathsf{id}_B}/\ \overline{g}$$

Identity and Composition

Composition is cut-elimination:

$$\frac{\alpha : S \underset{g}{\Longrightarrow} T \qquad \beta : T \underset{h}{\Longrightarrow} U}{\alpha \cdot \beta : S \underset{gh}{\Longrightarrow} U}$$

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Principal cuts:

$$\begin{array}{lcl} \delta^\eta \cdot \epsilon^\eta & \stackrel{\text{def}}{=} & (\delta\ \epsilon)^\eta \\ (\alpha.f) \cdot (f \backslash_h \beta) & \stackrel{\text{def}}{=} & \alpha \cdot \beta \\ (\alpha\ {}_g/\ \overline{f}) \cdot (\overline{f}.\beta) & \stackrel{\text{def}}{=} & \alpha \cdot \beta \end{array}$$

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(\alpha\ {}_g/\ \overline{f}) \cdot (\overline{f}.\beta) &\stackrel{\text{def}}{=} \alpha \cdot \beta
\end{aligned}$$

Commutative cuts:

$$\begin{aligned}
(\overline{f}.\alpha) \cdot \beta &\stackrel{\text{def}}{=} \overline{f}.(\alpha \cdot \beta) & \qquad \alpha \cdot (\beta.f) &\stackrel{\text{def}}{=} (\alpha \cdot \beta).f \\
(f \backslash_g \alpha) \cdot \beta &\stackrel{\text{def}}{=} f \backslash_{gh} (\alpha \cdot \beta) & \qquad \alpha \cdot \beta\ {}_h/\ \overline{f} &\stackrel{\text{def}}{=} (\alpha \cdot \beta)\ {}_{gh}/\ \overline{f}
\end{aligned}$$

Note: ambiguous cases up to equivalence.

Putting it all together

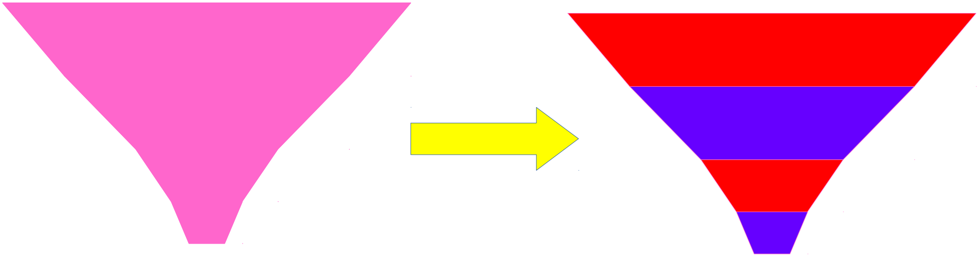
Let $\mathcal{B}\mathrm{if}(p)$ be the category whose objects are bifibrational formulas and whose arrows are $(\sim)$-equivalence classes of derivations, with composition defined by cut-elimination.

Let Λ_p be the functor $\mathcal{B}\mathrm{if}(p) \to \mathcal{C}$ sending $(S \sqsubset A)$ to A and $(\alpha : S \underset{f}{\Longrightarrow} T)$ to f.

Theorem. $\Lambda_p : \mathcal{B}\mathrm{if}(p) \to \mathcal{C}$ is the free bifibration on $p : \mathcal{D} \to \mathcal{C}$.

Multi-focusing

Rigid proof structure: invertible and non-invertible rules.



$\Gamma \vdash A$ $\qquad$ $\Gamma \vdash_{\mathrm{foc}} A$

Invertibility

Invertible, non-invertible rules (from conclusion to premises).

$$\frac{\alpha : S \underset{fg}{\Longrightarrow} T}{f\backslash_g\, \alpha : f^{+}\, S \underset{g}{\Longrightarrow} T}\ \mathsf{L}f^{+} \qquad \begin{tikzcd} \cdot \arrow[r, "fg"] \arrow[d, "f"] & \cdot \arrow[d, equal] \\ \cdot \arrow[r, "g"] & \cdot \end{tikzcd}$$

$$\frac{\alpha : S \underset{fg}{\Longrightarrow} T}{\alpha\ {}_f/\,\overline{g} : S \underset{f}{\Longrightarrow} g^{-}\ T}\ \mathsf{R}g^{-} \qquad \begin{tikzcd} \cdot \arrow[r, "fg"] \arrow[d, equal] & \cdot \\ \cdot \arrow[r, "f"] & \cdot \arrow[u, "g"'] \end{tikzcd}$$

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$$\frac{\alpha : S' \underset{e}{\Longrightarrow} S}{\alpha.f : S' \underset{ef}{\Longrightarrow} f^{+}\ S}\ \mathsf{R}f^{+} \qquad \begin{tikzcd} \cdot \arrow[r, "e"] \arrow[d, equal] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "ef"] & \cdot \end{tikzcd}$$

Polarized formulas

$$\pi, \rho, \sigma, \tau \quad ::= \quad (f_0, \dots, f_n) \qquad \text{non-empty sequence of composable arrows}$$

$$\begin{array}{rcl} S_{sa} & ::= & P_{sa} \mid N_{sa} \\ P_{sa} & ::= & X^\eta \mid \pi^+ N_{sa} \\ N_{sa} & ::= & X^\eta \mid \pi^- P_{sa} \end{array}$$

Multi-focused rules

$$\dfrac{\delta : X \to Y \in \mathcal{D} \qquad p(\delta) = f}{\delta^\eta : X^\eta \underset{f}{\Longrightarrow} Y^\eta}\ \delta^\eta$$

$$\dfrac{\alpha_m : N \underset{\pi f}{\Longrightarrow} P}{\pi \setminus_f \alpha_m : \pi^+ N \underset{f}{\Longrightarrow} P}\ \mathsf{L}\pi^+ \qquad \dfrac{\alpha_m : N \underset{\pi f \sigma}{\Longrightarrow} P}{\pi \setminus \underset{f}{\alpha_m} / \overline{\rho} : \pi^+ N \underset{f}{\Longrightarrow} \sigma^- P}\ \mathsf{L}\pi^+\mathsf{R}\sigma^- \qquad \dfrac{\alpha_m : N \underset{f\rho}{\Longrightarrow} P}{\alpha_{m\ f} / \overline{\rho} : N \underset{f}{\Longrightarrow} \rho^- P}\ \mathsf{R}\sigma^-$$

$$\dfrac{\alpha_m : P \underset{f}{\Longrightarrow} Q}{\overline{\pi}.\alpha_m : \pi^- P \underset{\pi f}{\Longrightarrow} Q}\ \mathsf{L}\pi^- \qquad \dfrac{\alpha_m : P \underset{f}{\Longrightarrow} N}{\overline{\pi}.\alpha_m.\sigma : \pi^- P \underset{\pi f \sigma}{\Longrightarrow} \sigma^+ N}\ \mathsf{L}\pi^-\mathsf{R}\sigma^+ \qquad \dfrac{\alpha_m : N \underset{f}{\Longrightarrow} M}{\alpha_m.\sigma : N \underset{f\sigma}{\Longrightarrow} \sigma^+ M}\ \mathsf{R}\sigma^+$$

$+$ normal forms via a confluent rewrite system, under a DPP condition

Now for some examples!

Example #1

Consider the following functor:

$$
\begin{tikzcd}
1 \arrow[d, "p"'] & 0 \arrow[d, dotted, no head] & \\
2 & 0 \arrow[r, "f"] & 1
\end{tikzcd}
$$

Puzzle: what is the free bififibration over p? Hmm...

Example #1

Consider the following functor:

$$\begin{tikzcd} 1 \arrow[d, "p"'] & 0 \arrow[d, dotted, no head] & \\ 2 & 0 \arrow[r, "f"] & 1 \end{tikzcd}$$

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Objects in $\mathcal{B}\mathrm{if}(p)_0$ are isomorphic to even-length sequences $S \equiv f^- \, f^+ \cdots f^- \, f^+ \, 0$

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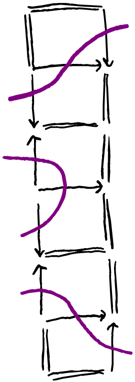
What are the arrows over 0?

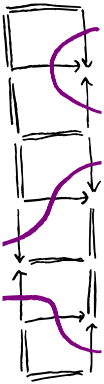
One morphism $2 \to 1$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{0 \xRightarrow[\mathrm{id}_0]{} 0}\ \mathit{id}}{0 \xRightarrow[f]{} f^+\ 0}\ \mathsf{R}f^+}{f^+\ 0 \xRightarrow[\mathrm{id}_1]{} f^+\ 0}\ \mathsf{L}f^+}{f^-\ f^+\ 0 \xRightarrow[f]{} f^+\ 0}\ \mathsf{L}f^-}{f^+\ f^-\ f^+\ 0 \xRightarrow[\mathrm{id}_1]{} f^+\ 0}\ \mathsf{L}f^+}{f^-\ f^+\ f^-\ f^+\ 0 \xRightarrow[f]{} f^+\ 0}\ \mathsf{L}f^-}{f^-\ f^+\ f^-\ f^+\ 0 \xRightarrow[\mathrm{id}_0]{} f^-\ f^+\ 0}\ \mathsf{R}f^-
$$

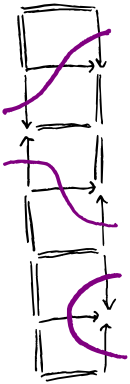


Two morphisms $1 \to 2$

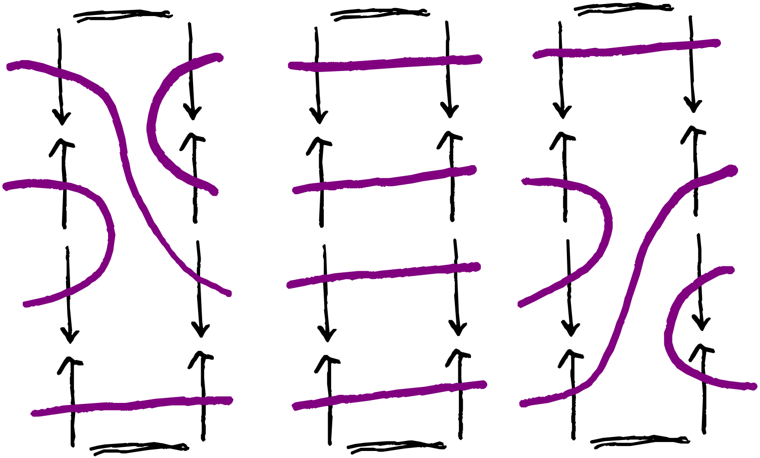
$$
\begin{array}{c}
\hline
0 \xRightarrow[\mathrm{id}_0]{} 0 \quad \mathrm{id}_0 \\
\hline
0 \xRightarrow[f]{} f^+\,0 \quad \mathrm{R}f^+ \\
\hline
0 \xRightarrow[\mathrm{id}_0]{} f^-\,f^+\,0 \quad \mathrm{R}f^- \\
\hline
0 \xRightarrow[f]{} f^+\,f^-\,f^+\,0 \quad \mathrm{R}f^+ \\
\hline
f^+\,0 \xRightarrow[\mathrm{id}_1]{} f^+\,f^-\,f^+\,0 \quad \mathrm{L}f^+ \\
\hline
f^-\,f^+\,0 \xRightarrow[f]{} f^+\,f^-\,f^+\,0 \quad \mathrm{L}f^- \\
\hline
f^-\,f^+\,0 \xRightarrow[\mathrm{id}_0]{} f^-\,f^+\,f^-\,f^+\,0 \quad \mathrm{R}f^-
\end{array}
$$



$$
\begin{array}{c}
\hline
0 \xRightarrow[\mathrm{id}_0]{} 0 \quad \mathrm{id}_0 \\
\hline
0 \xRightarrow[f]{} f^+\,0 \quad \mathrm{R}f^+ \\
\hline
f^+\,0 \xRightarrow[\mathrm{id}_1]{} f^+\,0 \quad \mathrm{L}f^+ \\
\hline
f^-\,f^+\,0 \xRightarrow[f]{} f^+\,0 \quad \mathrm{L}f^- \\
\hline
f^-\,f^+\,0 \xRightarrow[\mathrm{id}_0]{} f^-\,f^+\,0 \quad \mathrm{R}f^- \\
\hline
f^-\,f^+\,0 \xRightarrow[f]{} f^+\,f^-\,f^+\,0 \quad \mathrm{R}f^+ \\
\hline
f^-\,f^+\,0 \xRightarrow[\mathrm{id}_0]{} f^-\,f^+\,f^-\,f^+\,0 \quad \mathrm{R}f^-
\end{array}
$$



Three morphisms $2 \to 2$



$$f^- \; f^+ \; f^- \; f^+ \; 0 \underset{\mathrm{id}_0}{\Longrightarrow} f^- \; f^+ \; f^- \; f^+ \; 0$$

Punchline #1

Arrows $(f^- f^+)^m 0 \longrightarrow (f^- f^+)^n 0$ in $\mathcal{B}\mathrm{if}(p)_0$ correspond to monotone maps $\mathsf{m} \to \mathsf{n}$!

Indeed, the push-pull adjunction captures the adjunction

$$\begin{tikzcd} \Delta \arrow[rr, bend left, "f^+"] \arrow[rr, phantom, "\perp"] & & \Delta_\perp \arrow[ll, bend left, "f^-"] \end{tikzcd}$$

between the category Δ of finite ordinals and order-preserving maps, and the category $\Delta_\perp$ of non-empty finite ordinals and order-and-least-element-preserving maps.

Example #2

Now consider the following functor:

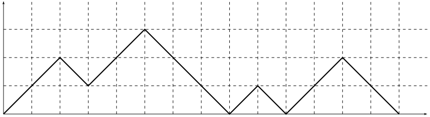
$$
\begin{tikzcd}
1 \arrow[d, "p"'] & 0 \arrow[d, dotted, no head] & & & \\
\mathbb{N} & 0 \arrow[r] & 1 \arrow[r] & 2 \arrow[r] & \dots
\end{tikzcd}
$$

Build the free bifibration $\mathcal{B}\mathrm{if}(p) \to \mathbb{N}$, and look at the fiber of 0.

Puzzle: what are its objects?

A category with Dyck walks as objects!

$f^- \, f^- \, f^+ \, f^+ \, f^- \, f^+ \, f^- \, f^- \, f^- \, f^+ \, f^+ \, f^- \, f^+ \, f^+ \, 0 \quad =$

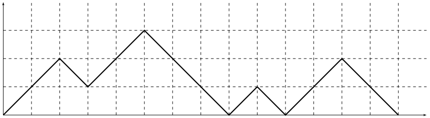


But what is a *morphism* of Dyck walks??

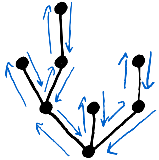
The $\mathcal{B}\mathrm{if}(-)$ construction gives an answer. Is it something natural/known?

Reconstructing the Batanin-Joyal category of trees

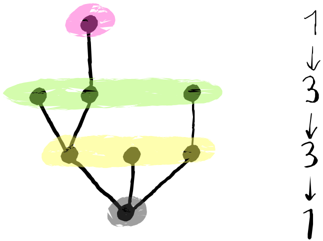
Dyck paths have a well-known, canonical bijection with (finite rooted plane) trees.



$\mapsto$



Trees may also be encoded as *functors* $T : \mathbb{N}^{op} \to \Delta$.



Reconstructing the Batanin-Joyal category of trees

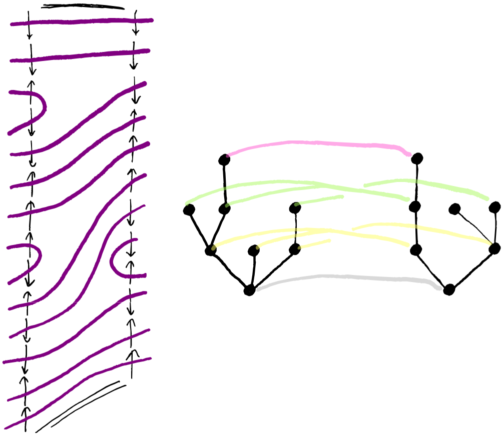
Consider *natural transformations* $\theta : S \Rightarrow T$.

$$
\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
S(2) \arrow[r, "\theta_2"] \arrow[d] & T(2) \arrow[d] \\
S(1) \arrow[r, "\theta_1"] \arrow[d] & T(1) \arrow[d] \\
S(0) = 1 \arrow[r, equal] & 1 = T(0)
\end{tikzcd}
$$

In other words, map nodes to nodes of the same height, respecting parents.

Punchline #2

Theorem: $\mathcal{B}\mathrm{if}(p : 1 \to \mathbb{N})_0 \cong \mathrm{PTree}$.



(More generally, $\mathcal{B}\mathrm{if}(p)_k \cong \mathrm{PTree}_k$ = category of finite rooted plane trees whose rightmost branch is pointed by a node of height k.)

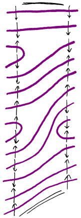
Summary

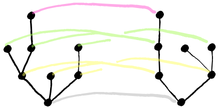
We have a clean and simple proof-theoretic construction of free bifibrations, with complentary algebraic & topological perspectives.

Normal forms characterized, under a DPP condition, by maximal multi-focusing.

Some surprisingly rich combinatorics emerges as if out of thin air.

$$\begin{tikzcd} 1 \arrow[d, "p"'] & 0 \arrow[d, dotted, no head] \\ \mathbb{N} & 0 \arrow[r] & 1 \arrow[r] & 2 \arrow[r] & \dots \end{tikzcd}$$





Thanks !

Questions ?

A category $\mathcal{C}$ is **factorization preordered** if for any diagram of composable arrows of the form

$$\begin{tikzcd} \cdot \arrow[r, shift left=2, "g"] \arrow[r, "f"] \arrow[r, shift right=2, "h"'] & \cdot \arrow[r, "k"] & {} \end{tikzcd}$$

if both $fg = fh$ and $gk = hk$ then necessarily $g = h$. Equivalently, $\mathcal{C}$ is factorization preordered just in case every commuting square has at most one diagonal filler:

$$\begin{tikzcd} \cdot \arrow[r, "f'"] \arrow[d, "f"', "k"] & \cdot \\ \cdot \arrow[r, "k'"'] \arrow[ur, dashed, "g=h"] & \cdot \end{tikzcd}$$