Focusing on program representations

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“Are these two proofs the same?”

“Are these two programs the same?”

(In this talk: propositional / simply typed setting.)
Section 1

Focusing
Focusing

Focusing is a technique from proof theory [Andreoli, 1992].

It studies **invertibility** of connectives to structure the search space.

Type theory perspective: canonical representations.

\[
\begin{align*}
    t & \approx_{\beta\eta} u \\
    \Rightarrow & \\
    t & \approx_\alpha u
\end{align*}
\]
Invertible vs. non-invertible rules. Positives vs. negatives.
\[
\begin{align*}
\Gamma \vdash A & \quad \Gamma, B \vdash C \quad \Rightarrow \quad \Gamma, A \rightarrow B \vdash C \\
\Gamma, A_1 \vdash C & \quad \Gamma, A_2 \vdash C \quad \Rightarrow \quad \Gamma, A_1 \times A_2 \vdash C \\
\Gamma, A \vdash B & \quad \Rightarrow \quad \Gamma \vdash A \rightarrow B \\
\Gamma \vdash A_1 & \quad \Gamma \vdash A_2 \quad \Rightarrow \quad \Gamma \vdash A_1 \times A_2 \\
\Gamma, 0 \vdash C & \quad \Rightarrow \quad \Gamma \vdash 1 \\
\Gamma, A_i \vdash C & \quad \Gamma, A_1 \vdash C, A_2 \vdash C \quad \Rightarrow \quad \Gamma, A_1 + A_2 \vdash C
\end{align*}
\]

Invertible vs. non-invertible rules. Positives vs. negatives.

\[
\begin{align*}
N, M & ::= A \rightarrow B \mid A \times B \mid 1 \\
A, B & ::= P \mid N \mid \alpha \\
P_a, Q_a & ::= P \mid \alpha \\
N_a, M_a & ::= N \mid \alpha
\end{align*}
\]
Invertible phase

If applied too early, non-invertible rules can ruin your proof.

Focusing restriction 1: invertible phases
Invertible rules must be applied as soon and as long as possible – and their order does not matter.
Invertible phase

$$\frac{\alpha + \beta \vdash \alpha}{\alpha + \beta \vdash \beta + \alpha}$$

If applied too early, non-invertible rules can ruin your proof.

**Focusing restriction 1: invertible phases**

Invertible rules must be applied as soon and as long as possible – and their order does not matter.

Imposing this restriction gives a single proof of \((\alpha \to \beta) \to (\alpha \to \beta)\) instead of two \((\lambda f. f \text{ and } \lambda f. \lambda x. f \ x)\).

After all invertible rules, negative context \(\Gamma_{na}\), positive goal \(P_a\).
Non-invertible phases

After all invertible rules, negative context, positive goal.
Only step forward: select a formula, apply some non-invertible rule on it.
Non-invertible phases

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Only step forward: select a formula, apply some non-invertible rule on it.

**Focusing restriction 2: non-invertible phase**

When a principal formula is selected for non-invertible rule, they should be applied as long as possible – until its polarity changes.
Non-invertible phases

After all invertible rules, negative context, positive goal.

Only step forward: select a formula, apply some non-invertible rule on it.

Focusing restriction 2: non-invertible phase

When a principal formula is selected for non-invertible rule, they should be applied as long as possible – until its polarity changes.

Completeness: this restriction preserves provability. Non-trivial!

Example of removed redundancy:

\[
\begin{align*}
\alpha_2, & \quad \beta_1 \vdash A \\
\alpha_2 \times \alpha_3, & \quad \beta_1 \vdash A \\
\alpha_2 \times \alpha_3, & \quad \beta_1 \times \beta_2 \vdash A \\
\alpha_1 \times \alpha_2 \times \alpha_3, & \quad \beta_1 \times \beta_2 \vdash A
\end{align*}
\]
This was focusing:
- invertible as long as a rule matches, until $\Gamma_{na} \vdash P_a$
- then pick a formula
- then non-invertible as long as a rule matches, until polarity change

Completeness:

\[ \Gamma \vdash A \quad \implies \quad \Gamma \vdash_{foc} A \]
a focused natural deduction

\[
\begin{align*}
N, M &::= A \to B \mid A \times B \mid 1 \\
A, B &::= P \mid N \mid \alpha \\
P_a, Q_a &::= P \mid \alpha \\
N_a, M_a &::= N \mid \alpha
\end{align*}
\]

\[
\begin{align*}
\Gamma_{na} &::= \emptyset \mid \Gamma_{na}, N_a
\end{align*}
\]

\[\Gamma_{na}; \Delta \vdash_{\text{inv}} A \text{ invertible phase (decomposes } \Delta, A)\]

\[\Gamma_{na} \vdash_{\text{foc}} P_a \text{ choice of focus}\]

\[\Gamma_{na}; N \Downarrow M_a \text{ non-invertible negative rules}\]

\[\Gamma_{na} \Uparrow P \text{ non-invertible positive rules}\]

(inspired by Brock-Nannestad and Schürmann [2010])
\[
\frac{\Gamma_{na}; \Delta, A \vdash_{\text{inv}} B}{\Gamma_{na}; \Delta \vdash_{\text{inv}} A \rightarrow B}
\]

\[
\frac{(\Gamma_{na}; \Delta \vdash_{\text{inv}} A_i)^i}{\Gamma_{na}; \Delta \vdash_{\text{inv}} A_1 \times A_2}
\]

\[
\frac{(\Gamma_{na}; \Delta, A_i \vdash_{\text{inv}} B)^i}{\Gamma_{na}; \Delta, A_1 + A_2 \vdash_{\text{inv}} B}
\]

\[
\frac{\Gamma_{na}; \Delta, 0 \vdash_{\text{inv}} A}{\Gamma_{na}; \Delta \vdash_{\text{inv}} 1}
\]

\[
\frac{\Gamma_{na} \uparrow P}{\Gamma_{na} \vdash_{\text{foc}} P}
\]

\[
\frac{\Gamma_{na}, N; N \downarrow \alpha}{\Gamma_{na}, N \vdash_{\text{foc}} \alpha}
\]

\[
\frac{\Gamma_{na}, N; N \downarrow P}{\Gamma_{na}, P \vdash_{\text{inv}} Q_a}
\]

\[
\frac{\Gamma_{na}, N; N \downarrow P}{\Gamma_{na}, N \vdash_{\text{foc}} Q_a}
\]

\[
\frac{\Gamma_{na} \uparrow A_i}{\Gamma_{na} \uparrow A_1 + A_2}
\]

\[
\frac{\Gamma_{na} \uparrow \alpha}{\Gamma_{na} \uparrow \alpha}
\]

\[
\frac{\Gamma_{na}; \emptyset \vdash_{\text{inv}} N}{\Gamma_{na} \uparrow N}
\]

\[
\frac{\Gamma_{na}; N \downarrow N}{\Gamma_{na}; N \downarrow A \times A_2}
\]

\[
\frac{\Gamma_{na}; N \downarrow A_i}{\Gamma_{na}; N \downarrow A_i}
\]

\[
\frac{\Gamma_{na}; N \downarrow A \rightarrow B}{\Gamma_{na} \uparrow A}
\]

\[
\frac{\Gamma_{na}; N \downarrow B}{\Gamma_{na}; N \downarrow B}
\]

(some simplifications, see Scherer [2016] for full details)
Section 2

Focused $\lambda$-calculus
\(\beta\)-normal forms (negative)

\(\beta\)-short normal forms:

\[
\pi_1 \left( t, u \right) = t
\]

\[
v, w \ ::= \ \lambda x. v \mid (v, w) \mid n
\]

\[
n, m \ ::= \ \pi_i n \mid n v \mid x
\]
**β-normal forms (negative)**

**β-short normal forms:**

\[ \pi_1 (t, u) = t \]

\[ v, w ::= \lambda x. v \mid (v, w) \mid n \]

\[ n, m ::= \pi_i n \mid n v \mid x \]

**β-short η-long:**

\[ (y : \alpha \rightarrow \beta) = \lambda x : \alpha. (y x : \beta) \]
**β-normal forms (negative)**

**β-short normal forms:**

\[
\pi_1 (t, u) = t
\]

\[
v, w ::= \lambda x. v \mid (v, w) \mid n
\]

\[
n, m ::= \pi_i n \mid n v \mid x
\]

**β-short η-long:**

\[
(y : \alpha \to \beta) = \lambda x : \alpha. (y x : \beta)
\]

\[
v, w ::= \lambda x. v \mid (v, w) \mid (n : \alpha)
\]

\[
n, m ::= \pi_i n \mid n v \mid x
\]
What about sums?

\[
\begin{align*}
v, w & ::= \lambda x. v \mid (v, w) \mid \sigma_i \ v \mid (n : \alpha) \\
n, m & ::= \pi_i n \mid n \ v \mid \left( \text{match } n \text{ with } \begin{array}{l}
\sigma_1 y_1 \rightarrow v_1 \\
\sigma_2 y_2 \rightarrow v_2 
\end{array} \right) \mid x
\end{align*}
\]

Does not work:

\[
\left( \begin{array}{ll}
\text{match } n \text{ with } & \\
\sigma_1 y_1 \rightarrow \lambda z. v_1 \\
\sigma_2 y_2 \rightarrow \lambda z. v_2
\end{array} \right) v
\]
Focusing to the rescue

\[
v, w ::= \lambda x. v \mid (v, w) \mid (n : \alpha) \\
n, m ::= \pi_i n \mid n \; v \mid x
\]

\[
\Gamma_{na}; \Delta \vdash_{\text{inv}} \nu : A \\n\Gamma_{na} \vdash n \downarrow N_a \\
\Gamma_{na} \vdash p \uparrow P_a \\
\Gamma_{na} \vdash_{\text{foc}} f : A
\]

\[
\begin{align*}
\nu, w & ::= \lambda x. v \mid (v, w) \mid () \\
& \quad \mid \text{absurd}(x) \mid \text{match } x \text{ with } \\
& \quad \mid (\Gamma_{na} \vdash f : P_a)
\end{align*}
\]

\[
\begin{align*}
n, m & ::= \pi_i n \mid n \; p \mid x \\
p, q & ::= \sigma_i p \mid (v : N_a)
\end{align*}
\]

\[
\begin{align*}
f & ::= \text{let } x = (n : P) \text{ in } v \\
& \quad \mid (n : \alpha) \mid (p : P)
\end{align*}
\]

(See also Munch-Maccagnoni [2013])
Completeness of focusing

Logic:

\[ \Gamma \vdash A \quad \Rightarrow \quad \Gamma \vdash_{foc} A \]
Completeness of focusing

Logic:

\[ \Gamma \vdash A \implies \Gamma \vdash_{\text{foc}} A \]

Programming:

\[ \Gamma \vdash t : A \implies \exists v, \Gamma \vdash_{\text{foc}} v : A \]

\[ v \approx_{\beta \eta} t \]
Canonicity

Focused normal forms are canonical for the impure $\lambda$-calculus.

Proof in Zeilberger [2009], using ideas from Girard’s ludics.
Canonicity

Focused normal forms are canonical for the impure $\lambda$-calculus.

Proof in Zeilberger [2009], using ideas from Girard’s ludics.

Not canonical for the pure calculus.

\[
\text{let } x = n \text{ in } C [\text{let } x' = n' \text{ in } v] \\
\text{let } x' = n' \text{ in } C [\text{let } x = n \text{ in } v]
\]
Canonicity

Focused normal forms are canonical for the impure $\lambda$-calculus.

Proof in Zeilberger [2009], using ideas from Girard’s ludics.

Not canonical for the pure calculus.

\[
\text{let } x = n \text{ in } C \left[ \text{let } x' = n' \text{ in } v \right]
\]

\[
\text{let } x' = n' \text{ in } C \left[ \text{let } x = n \text{ in } v \right]
\]

Solution: “saturation” [Scherer, 2017]

\[
f ::= \text{let } \bar{x} = \bar{n} \text{ in } v \mid (n : \alpha) \mid (p : P)
\]

inspired by multi-focusing [Chaudhuri, Miller, and Saurin, 2008].
Multi-focusing in one slide

if C does not depend on B...
Multi-focusing in one slide

if C does not depend on B...
Applications of focusing and canonicity

A clean way to extend our understanding to positives (+, 0).

- evaluation order in presence of effects
- which types have a unique inhabitant?
- decidability of equivalence
- Böhm separation results: contextual and ($\beta\eta$) coincide
- $\lambda$-definability?
- (your result here!)


