Focusing on program representations

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Intro

"Are these two proofs the same?"

$$\simeq$$

"Are these two programs the same?"

(In this talk: propositional / simply typed setting.)

Section 1

Focusing

Focusing

Focusing is a technique from proof theory [Andreoli, 1992].

It studies **invertibility** of connectives
to structure the search space.

Type theory perspective: canonical representations.

$$t \approx_{\beta\eta} u \quad \overset{?}{\Longrightarrow} \quad t \approx_{\alpha} u$$

$$\frac{\Gamma \vdash \underline{A} \qquad \Gamma, \underline{B} \vdash C}{\Gamma, \underline{A \to B} \vdash C}\ -$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}$$

$$\frac{\Gamma, \underline{A_i} \vdash C}{\Gamma, \underline{A_1 \times A_2} \vdash C}\ -$$

$$\frac{\Gamma \vdash A_1 \qquad \Gamma \vdash A_2}{\Gamma \vdash A_1 \times A_2}$$

$$\frac{\Gamma, A_1 \vdash C \qquad \Gamma, A_2 \vdash C}{\Gamma, A_1 + A_2 \vdash C}$$

$$\frac{\Gamma \vdash \underline{A_i}}{\Gamma \vdash \underline{A_1 + A_2}}\ +$$

$$\frac{}{\Gamma, 0 \vdash C}\ +$$

$$\frac{}{\Gamma \vdash 1}\ -$$

Invertible vs. non-invertible rules. Positives vs. negatives.

$$\frac{\Gamma \vdash \underline{A} \qquad \Gamma, \underline{B} \vdash C}{\Gamma, \underline{A \to B} \vdash C}\;- \qquad\qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}$$

$$\frac{\Gamma, \underline{A_i} \vdash C}{\Gamma, \underline{A_1 \times A_2} \vdash C}\;- \qquad\qquad \frac{\Gamma \vdash A_1 \qquad \Gamma \vdash A_2}{\Gamma \vdash A_1 \times A_2}$$

$$\frac{\Gamma, A_1 \vdash C \qquad \Gamma, A_2 \vdash C}{\Gamma, A_1 + A_2 \vdash C} \qquad\qquad \frac{\Gamma \vdash \underline{A_i}}{\Gamma \vdash \underline{A_1 + A_2}}\;+$$

$$\frac{}{\Gamma, 0 \vdash C}\;+ \qquad\qquad \frac{}{\Gamma \vdash 1}\;-$$

Invertible vs. non-invertible rules. Positives vs. negatives.

$$N, M ::= A \to B \mid A \times B \mid 1 \qquad\qquad P, Q ::= A + B \mid 0$$

$$A, B ::= P \mid N \mid \alpha \qquad\qquad P_{\mathrm{a}}, Q_{\mathrm{a}} ::= P \mid \alpha \qquad\qquad N_{\mathrm{a}}, M_{\mathrm{a}} ::= N \mid \alpha$$

Invertible phase

$$\dfrac{\dfrac{?}{\alpha + \beta \vdash \alpha}}{\alpha + \beta \vdash \beta + \alpha}$$

If applied too early, non-invertible rules can ruin your proof.

Focusing restriction 1: invertible phases

Invertible rules must be applied as soon and as long as possible
– and their order does not matter.

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Focusing restriction 1: invertible phases

Invertible rules must be applied as soon and as long as possible
– and their order does not matter.

Imposing this restriction gives a single proof of $(\alpha \to \beta) \to (\alpha \to \beta)$ instead of two ($\lambda f.\, f$ and $\lambda f.\, \lambda x.\, f\ x$).

After all invertible rules, negative context Γ_{na}, positive goal P_{a}.

Non-invertible phases

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Only step forward: select a formula, apply some non-invertible rule on it.

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Focusing restriction 2: non-invertible phase

When a principal formula is selected for non-invertible rule, they should be applied as long as possible – until its polarity changes.

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Focusing restriction 2: non-invertible phase

When a principal formula is selected for non-invertible rule, they should be applied as long as possible – until its polarity changes.

Completeness: this restriction preserves provability. **Non-trivial !**
Example of removed redundancy:

$$\dfrac{\dfrac{\dfrac{\alpha_2, \quad \beta_1 \vdash A}{\alpha_2 \times \alpha_3, \quad \beta_1 \vdash A}}{\alpha_2 \times \alpha_3, \quad \beta_1 \times \beta_2 \vdash A}}{\alpha_1 \times \alpha_2 \times \alpha_3, \beta_1 \times \beta_2 \vdash A}$$

This was focusing:

- invertible as long as a rule matches, until $\Gamma_{\mathsf{na}} \vdash P_{\mathsf{a}}$
- then pick a formula
- then non-invertible as long as a rule matches, until polarity change

Completeness:

$$\Gamma \vdash A \qquad \Longrightarrow \qquad \Gamma \vdash_{\texttt{foc}} A$$

a focused natural deduction

$$N, M ::= A \to B \mid A \times B \mid 1 \qquad P, Q ::= A + B \mid 0$$

$$A, B ::= P \mid N \mid \alpha \qquad P_{\mathsf{a}}, Q_{\mathsf{a}} ::= P \mid \alpha \qquad N_{\mathsf{a}}, M_{\mathsf{a}} ::= N \mid \alpha$$

$$\Gamma_{\mathsf{na}} ::= \emptyset \mid \Gamma_{\mathsf{na}}, N_{\mathsf{a}}$$

$\Gamma_{\mathsf{na}}; \Delta \vdash_{\mathsf{inv}} A$ invertible phase (decomposes Δ, A)

$\Gamma_{\mathsf{na}} \vdash_{\mathsf{foc}} P_{\mathsf{a}}$ choice of focus

$\Gamma_{\mathsf{na}}; N \Downarrow M_{\mathsf{a}}$ non-invertible negative rules

$\Gamma_{\mathsf{na}} \Uparrow P$ non-invertible positive rules

(inspired by Brock-Nannestad and Schürmann [2010])

$$\frac{\Gamma_{na}; \Delta, A \vdash_{inv} B}{\Gamma_{na}; \Delta \vdash_{inv} A \to B} \qquad \frac{(\Gamma_{na}; \Delta \vdash_{inv} A_i)^i}{\Gamma_{na}; \Delta \vdash_{inv} A_1 \times A_2} \qquad \frac{(\Gamma_{na}; \Delta, A_i \vdash_{inv} B)^i}{\Gamma_{na}; \Delta, A_1 + A_2 \vdash_{inv} B}$$

$$\frac{}{\Gamma_{na}; \Delta, 0 \vdash_{inv} A} \qquad \frac{}{\Gamma_{na}; \Delta \vdash_{inv} 1} \qquad \frac{\Gamma_{na}, \Gamma'_{na} \vdash_{foc} P_a}{\Gamma_{na}; \Gamma'_{na} \vdash_{inv} P_a}$$

$$\frac{\Gamma_{na} \Uparrow P}{\Gamma_{na} \vdash_{foc} P} \qquad \frac{\Gamma_{na}, N; N \Downarrow \alpha}{\Gamma_{na}, N \vdash_{foc} \alpha} \qquad \frac{\Gamma_{na}, N; N \Downarrow P \qquad \Gamma_{na}; P \vdash_{inv} Q_a}{\Gamma_{na}, N \vdash_{foc} Q_a}$$

$$\frac{\Gamma_{na} \Uparrow A_i}{\Gamma_{na} \Uparrow A_1 + A_2} \qquad \frac{}{\Gamma_{na}, \alpha \Uparrow \alpha} \qquad \frac{\Gamma_{na}; \emptyset \vdash_{inv} N}{\Gamma_{na} \Uparrow N}$$

$$\frac{}{\Gamma_{na}; N \Downarrow N} \qquad \frac{\Gamma_{na}; N \Downarrow A_1 \times A_2}{\Gamma_{na}; N \Downarrow A_i} \qquad \frac{\Gamma_{na}; N \Downarrow A \to B \qquad \Gamma_{na} \Uparrow A}{\Gamma_{na}; N \Downarrow B}$$

(some simplifications, see Scherer [2016] for full details)

Section 2

Focused λ-calculus

β-normal forms (negative)

β-short normal forms:

$$\pi_1\ (t, u) = t$$

$$\begin{array}{rcl} v, w & ::= & \lambda x.\, v \mid (v, w) \mid n \\ n, m & ::= & \pi_i\ n \mid n\ v \mid x \end{array}$$

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What about sums?

$$v, w \;::=\; \lambda x.\, v \mid (v, w) \mid \sigma_i\ v \mid (n : \alpha)$$

$$n, m \;::=\; \pi_i\ n \mid n\ v \mid \left(\texttt{match}\ n\ \texttt{with}\ \left|\begin{array}{l} \sigma_1\ y_1 \to v_1 \\ \sigma_2\ y_2 \to v_2 \end{array}\right.\right) \mid x$$

Does not work:

$$\left(\begin{array}{l}\texttt{match}\ n\ \texttt{with} \\ \left|\begin{array}{l} \sigma_1\ y_1 \to \lambda z.\, v_1 \\ \sigma_2\ y_2 \to \lambda z.\, v_2 \end{array}\right.\end{array}\right) v \qquad\qquad \begin{array}{l}\texttt{match}\ n\ \texttt{with} \\ \left|\begin{array}{l} \sigma_1\ x \to \sigma_2\ x \\ \sigma_2\ x \to \sigma_1\ x \end{array}\right.\end{array}$$

Focusing to the rescue

$$
\begin{array}{rcl}
v, w & ::= & \lambda x.\, v \mid (v, w) \mid (n : \alpha) \\
n, m & ::= & \pi_i\ n \mid n\ v \mid x
\end{array}
$$

$$\Downarrow$$

$$
\begin{array}{lll}
\Gamma_{\mathrm{na}}; \Delta \vdash_{\mathrm{inv}} v : A & v, w ::= & \lambda x.\, v \mid (v, w) \mid () \\
& & \mid \mathtt{absurd}(x) \mid \mathtt{match}\ x\ \mathtt{with} \left| \begin{array}{l} \sigma_1\ y_1 \to v_1 \\ \sigma_2\ y_2 \to v_2 \end{array} \right. \\
& & \mid (\Gamma_{\mathrm{na}} \vdash f : P_{\mathrm{a}}) \\
\\
\Gamma_{\mathrm{na}} \vdash n \Downarrow N_{\mathrm{a}} & n, m ::= & \pi_i\ n \mid n\ p \mid x \\
\Gamma_{\mathrm{na}} \vdash p \Uparrow P_{\mathrm{a}} & p, q ::= & \sigma_i\ p \mid (v : N_{\mathrm{a}}) \\
\\
\Gamma_{\mathrm{na}} \vdash_{\mathrm{foc}} f : A & f ::= & \mathtt{let}\ x = (n : P)\ \mathtt{in}\ v \\
& & \mid (n : \alpha) \mid (p : P)
\end{array}
$$

(See also Munch-Maccagnoni [2013])

Completeness of focusing

Logic:

$$\Gamma \vdash A \qquad \Longrightarrow \qquad \Gamma \vdash_{\texttt{foc}} A$$

Completeness of focusing

Logic:

$$\Gamma \vdash A \qquad \Longrightarrow \qquad \Gamma \vdash_{\texttt{foc}} A$$

Programming:

$$\Gamma \vdash t : A \qquad \Longrightarrow \qquad \exists v, \begin{array}{c} \Gamma \vdash_{\texttt{foc}} v : A \\ v \approx_{\beta\eta} t \end{array}$$

Canonicity

Focused normal forms are canonical for the impure λ-calculus.

Proof in Zeilberger [2009], using ideas from Girard's ludics.

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$$\texttt{let}\ x = n\ \texttt{in}\ C\left[\texttt{let}\ x' = n'\ \texttt{in}\ v\right]$$

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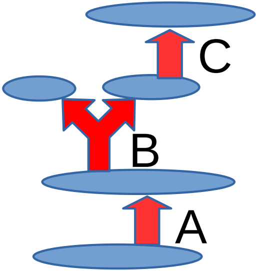
Solution: "saturation" [Scherer, 2017]

$$f \quad ::= \quad \texttt{let}\ \bar{x} = \bar{n}\ \texttt{in}\ v \mid (n : \alpha) \mid (p : P)$$

inspired by multi-focusing [Chaudhuri, Miller, and Saurin, 2008].

Multi-focusing in one slide



if C does not depend on B...

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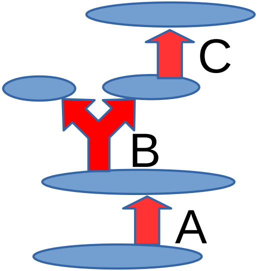

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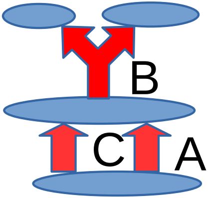

Applications of focusing and canonicity

A clean way to extend our understanding to positives ($+$, 0).

- evaluation order in presence of effects
- which types have a unique inhabitant?
- decidability of equivalence
- Böhm separation results: contextual and ($\beta\eta$) coincide
- λ-definability?
- (your result here!)

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