Design

- $i\text{ML}^F$: an implicitly-typed extension of System F
- Types explained
- $e\text{ML}^F$: an explicitly-typed version of $i\text{ML}^F$

Uses and Implementation

- Examples
- Type inference
- Restrictions and extensions
MLF for Everyone
(Users, Implementers, and Designers)

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ML Workshop

(Based on joint work with Didier Le Botlan and Boris Yakobowski)
Simple to use
Expressive
Great success
Happy days
F

M₅
Expressive

Simple extension of ML
Simplification

Even used in full scale languages such as Scala.

Full type inference is undecidable

Full type annotations are obfuscating
Outline

1 Design
   - $iML^F$: an implicitly-typed extension of System F
   - Types explained
   - $eML^F$: an explicitly-typed version of $iML^F$

2 Uses and Implementation
   - Examples
   - Type inference
   - Restrictions and extensions
A universal type system

**Explicit System F:**

\[
\begin{align*}
\text{VAR} & \quad z : \tau \in \Gamma \\
& \quad \Gamma \vdash z : \tau \\
\text{APP} & \quad \Gamma \vdash a_1 : \tau_2 \to \tau_1 \quad \Gamma \vdash a_2 : \tau_2 \\
& \quad \Gamma \vdash a_1 \ a_2 : \tau_1 \\
\text{FUN} & \quad \Gamma, x : \tau_0 \vdash a : \tau \\
& \quad \Gamma \vdash \lambda(x : \tau_0) \ a : \tau_0 \to \tau \\
\text{GEN} & \quad \Gamma, \alpha \vdash a : \tau_0 \\
& \quad \Gamma \vdash \wedge\alpha. \ a : \forall(\alpha) \ \tau_0 \\
\text{UNGEN} & \quad \Gamma \vdash a : \forall(\alpha) \ \tau \\
& \quad \Gamma \vdash a \ \tau : \tau_0[\alpha \leftarrow \tau]
\end{align*}
\]
A universal type system

Implicit System F:

\[
\begin{align*}
\text{VAR} & \\
\Gamma, z : \tau & \in \Gamma \\
\hline
\Gamma & \vdash z : \tau
\end{align*}
\]

\[
\begin{align*}
\text{APP} & \\
\Gamma & \vdash a_1 : \tau_2 \rightarrow \tau_1 \\
\Gamma & \vdash a_2 : \tau_2 \\
\hline
\Gamma & \vdash a_1 a_2 : \tau_1
\end{align*}
\]

\[
\begin{align*}
\text{GEN} & \\
\Gamma, \alpha & \vdash a : \tau_0 \\
\hline
\Gamma & \vdash a : \forall (\alpha) \tau_0
\end{align*}
\]

\[
\begin{align*}
\text{UNGEN} & \\
\Gamma & \vdash a : \forall (\alpha) \tau \\
\hline
\Gamma & \vdash a : \tau_0[\alpha \leftarrow \tau]
\end{align*}
\]

\[
\begin{align*}
\text{FUN} & \\
\Gamma, x : \tau_0 & \vdash a : \tau \\
\hline
\Gamma & \vdash \lambda (x) \hspace{2pt} a : \tau_0 \rightarrow \tau
\end{align*}
\]
A universal type system

**Implicit System F:**

**VAR**

\[ z : \tau \in \Gamma \]

\[ \Gamma \vdash z : \tau \]

**APP**

\[ \Gamma \vdash a_1 : \tau_2 \to \tau_1 \quad \Gamma \vdash a_2 : \tau_2 \]

\[ \Gamma \vdash a_1 \ a_2 : \tau_1 \]

**FUN**

\[ \Gamma, \ x : \tau_0 \vdash a : \tau \]

\[ \Gamma \vdash \lambda (x \quad ) \ a : \tau_0 \to \tau \]

**GEN**

\[ \Gamma, \alpha \vdash a : \tau_0 \]

\[ \Gamma \vdash a : \forall (\alpha) \ \tau_0 \]

**INST**

\[ \forall (\bar{\alpha}) \ \tau_0 \leq \tau_0[\bar{\alpha} \leftarrow \bar{\tau}] \]

**SUB**

\[ \Gamma \vdash a : \tau_1 \quad \tau_1 \leq \tau_2 \]

\[ \Gamma \vdash a : \tau_2 \]
A universal type system

Implicit System F:

\[
\begin{align*}
\text{VAR} & \quad z : \tau \in \Gamma \\
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\text{APP} & \quad \Gamma \vdash a_1 : \tau_2 \rightarrow \tau_1 \quad \Gamma \vdash a_2 : \tau_2 \\
\Gamma \vdash a_1 a_2 : \tau_1 \\
\text{FUN} & \quad \Gamma, x : \tau_0 \vdash a : \tau \\
\Gamma \vdash \lambda(x) \ a : \tau_0 \rightarrow \tau \\
\text{GEN} & \quad \Gamma, \alpha \vdash a : \tau_0 \\
\Gamma \vdash a : \forall(\alpha) \tau_0 \\
\text{INST} & \quad \beta \notin \text{ftv}(\forall(\bar{\alpha}) \ \bar{\tau}_0) \\
\forall(\bar{\alpha}) \ \tau_0 \leq \forall(\bar{\beta}) \ \tau_0[\bar{\alpha} \leftarrow \bar{\tau}] \\
\text{SUB} & \quad \Gamma \vdash a : \tau_1 \quad \tau_1 \leq \tau_2 \\
\Gamma \vdash a : \tau_2
\end{align*}
\]
A universal type system

**Implicit System F:**

\[
\begin{align*}
\text{VAR} & \quad z : \tau \in \Gamma \\
& \implies \Gamma \vdash z : \tau \\
\text{APP} & \quad \Gamma \vdash a_1 : \tau_2 \rightarrow \tau_1 \quad \Gamma \vdash a_2 : \tau_2 \\
& \implies \Gamma \vdash a_1 \ a_2 : \tau_1 \\
\text{FUN} & \quad \Gamma, x : \tau_0 \vdash a : \tau \\
& \implies \Gamma \vdash \lambda (x) \ a : \tau_0 \rightarrow \tau \\
\text{GEN} & \quad \Gamma, \alpha \vdash a : \tau_0 \\
& \implies \Gamma \vdash a : \forall (\alpha) \ \tau_0 \\
\text{INST} & \quad \begin{cases} 
\bar{\beta} \not\in \text{ftv}(\forall (\bar{\alpha}) \ \tau_0) \\
\forall (\bar{\alpha}) \ \tau_0 \leq \forall (\bar{\beta}) \ \tau_0[\bar{\alpha} \leftarrow \bar{\tau}] 
\end{cases} \\
& \implies \Gamma \vdash a : \tau_2 \\
\text{SUB} & \quad \Gamma \vdash a : \tau_1 \quad \tau_1 \leq \tau_2 \\
& \implies \Gamma \vdash a : \tau_2 \\
\end{align*}
\]

Add a construction for local bindings (perhaps derivable):

\[
\begin{align*}
\text{LET} & \quad \Gamma \vdash a_1 : \tau_1 \\
& \quad \Gamma, x : \tau_1 \vdash a_2 : \tau \\
& \implies \Gamma \vdash \text{let } x = a_1 \ \text{in } a_2 : \tau 
\end{align*}
\]
A universal type system

**Implicit System F:**

\[
\begin{align*}
\text{VAR} & \quad z : \tau \in \Gamma \\
\Gamma \vdash z : \tau
\end{align*}
\]

\[
\begin{align*}
\text{APP} & \quad \Gamma \vdash a_1 : \sigma_1 \\
 & \quad \Gamma \vdash a_2 : \sigma_2 \\
\Gamma \vdash a_1 a_2 : \sigma_0
\end{align*}
\]

**Logical, canonical presentation of typing rules**

- Covers many variations: \( F, \text{ML}, F^n, F_\leq, \ldots \)
  - Vary the set of types.
  - Vary the instance relation between types.
- For ML, **just restrict** types to ML types.

Add a construction for local bindings (perhaps derivable):

\[
\begin{align*}
\text{LET} & \quad \Gamma \vdash a_1 : \tau_1 \\
 & \quad \Gamma, x : \tau_1 \vdash a_2 : \sigma \\
\Gamma \vdash \text{let } x = a_1 \text{ in } a_2 : \sigma
\end{align*}
\]
A universal type system

**Implicit System F:**

\[
\begin{align*}
\text{VAR} & \quad \frac{z : \tau \in \Gamma}{\Gamma \vdash z : \tau} \\
\text{APP} & \quad \frac{\Gamma \vdash \alpha : \tau_0}{\Gamma \vdash a : \forall(\alpha) \tau_0}
\end{align*}
\]

**Logical, canonical presentation of typing rules**

- Covers many variations: F, ML, F\(^n\), F\(_\leq\), ...
  - Vary the set of types.
  - Vary the instance relation between types.

**For ML, just restrict** types to ML types.

**Do never change the typing rules!**

Add a construction for local bindings (perhaps derivable):

\[
\text{LET} \quad \frac{\Gamma \vdash a_1 : \tau_1 \quad \Gamma, x : \tau_1 \vdash a_2 : \tau}{\Gamma \vdash \text{let } x = a_1 \text{ in } a_2 : \tau}
\]
Type inference is undecidable — in System F

Of course, we must

- Use type annotations on function parameters in some cases.

When?

- Always?
  - too many annotations are obfuscating.
- Alleviate some annotations by local type inference?
  - intuitive and fragile (to program transformations).
- When parameters have polymorphic types?
  - still too many bothersome type annotations.

Are polymorphic types less important than monomorphic ones?
Type inference is undecidable — in System F

Of course, we must

- Use type annotations on function parameters in some cases.

When?

- Always?
  - too many annotations are obfuscating.
- Alleviate some annotations by local type inference?
  - unintuitive and fragile (to program transformations).
- When parameters have polymorphic types?
  - still too many bothersome type annotations.

Are polymorphic types less important than monomorphic ones?

Our choice

- When (and only when) parameters are used polymorphically.
Lack of principal types for applications

The example of choice

\[
\text{let } \text{choice} = \lambda(x) \lambda(y) \text{ if } \text{true} \text{ then } x \text{ else } y : \forall \beta \cdot \beta \rightarrow \beta \rightarrow \beta \\
\text{let } \text{id} = \lambda(z) z : \forall(\alpha) \alpha \rightarrow \alpha \\
\text{choice id :}
\]
Lack of principal types for applications

The example of choice

let choice = \lambda(x) \lambda(y) \text{if true then } x \text{ else } y : \forall \beta \cdot \beta \to \beta \to \beta

let id = \lambda(z) z : \forall(\alpha) \alpha \to \alpha

choice id : \begin{cases} \forall(\alpha) (\alpha \to \alpha) \to (\alpha \to \alpha) \\ (\forall(\alpha) \alpha \to \alpha) \to (\forall(\alpha) \alpha \to \alpha) \end{cases}
Lack of principal types for applications

The example of choice

let choice = \( \lambda(x) \lambda(y) \) if true then \( x \) else \( y \) : \( \forall \beta \cdot \beta \to \beta \to \beta \)
let id = \( \lambda(z) z \) : \( \forall(\alpha) \, \alpha \to \alpha \)

choice id : \[
\begin{align*}
\forall(\alpha) \, (\alpha \to \alpha) & \to (\alpha \to \alpha) \\
(\forall(\alpha) \, \alpha \to \alpha) & \to (\forall(\alpha) \, \alpha \to \alpha)
\end{align*}
\]

No better choice in F!
Lack of principal types for applications

The example of choice

let choice = \( \lambda(x) \lambda(y) \) if true then \( x \) else \( y \) : \( \forall \beta \cdot \beta \to \beta \to \beta \)

let id = \( \lambda(z) z \) : \( \forall(\alpha) \alpha \to \alpha \)

choice id : \[
\begin{aligned}
\forall(\alpha) (\alpha \to \alpha) \to (\alpha \to \alpha) \\
(\forall(\alpha) \alpha \to \alpha) \to (\forall(\alpha) \alpha \to \alpha)
\end{aligned}
\]

No better choice in F!

The problem is serious and inherent

- Follows from rules \texttt{Inst}, \texttt{Gen}, and \texttt{App}.
- Should values be kept as polymorphic or as instantiated as possible?
- A type inference system can do both, but cannot choose.
Lack of principal types for applications

The example of choice

\[\text{let } \text{choice} = \lambda(x) \lambda(y) \text{ if true then } x \text{ else } y : \forall \beta \cdot \beta \rightarrow \beta \rightarrow \beta\]

\[\text{let } \text{id} = \lambda(z) z : \forall(\alpha) \alpha \rightarrow \alpha\]

\[\text{choice id} : \begin{cases} 
\forall(\alpha) (\alpha \rightarrow \alpha) \rightarrow (\alpha \rightarrow \alpha) \\
(\forall(\alpha) \alpha \rightarrow \alpha) \rightarrow (\forall(\alpha) \alpha \rightarrow \alpha) 
\end{cases}\]

The solution in iMLF:

\[\text{choice id} : \forall(\beta \geq \forall(\alpha) \alpha \rightarrow \alpha) \beta \rightarrow \beta\]
Lack of principal types for applications

The example of choice

let choice = \lambda(x) \lambda(y) \text{if } true \text{ then } x \text{ else } y : \forall \beta \cdot \beta \rightarrow \beta \rightarrow \beta
let id = \lambda(z) z : \forall(\alpha) \alpha \rightarrow \alpha

choice id : \begin{cases} \forall(\alpha)(\alpha \rightarrow \alpha) \rightarrow (\alpha \rightarrow \alpha) \\ \forall(\alpha) \alpha \rightarrow \alpha \rightarrow (\forall(\alpha) \alpha \rightarrow \alpha) \end{cases}

The solution in iMLF:

choice id : \forall(\beta \geq \forall(\alpha) \alpha \rightarrow \alpha) \beta \rightarrow \beta

\ll \begin{cases} \beta \rightarrow \beta \left[ \beta \leftarrow \forall(\alpha) \alpha \rightarrow \alpha \right] \\ \forall(\alpha) (\beta \rightarrow \beta) \left[ \beta \leftarrow \alpha \rightarrow \alpha \right] \end{cases}
The definition of $iML^F$

**Types are stratified**

\[
\sigma ::= \tau \in F \\
\quad | \quad \forall (\alpha \geq \sigma) \sigma
\]

We can see and explain types by $\leq_F$-closed sets of System-F types:

\[
\{\tau\} \triangleq \{\tau' \mid \tau \leq_F \tau'\} \\
\{\forall (\alpha \geq \sigma) \sigma'\} \triangleq \left\{\forall (\overline{\beta}) \tau'[\alpha \leftarrow \tau] \mid \wedge \left(\begin{array}{l}
\tau \in \{\sigma\} \\
\tau' \in \{\sigma'\} \\
\overline{\beta} \notin ftv(\forall (\alpha \geq \sigma) \sigma')
\end{array}\right)\right\}
\]

**Type instance $\leq_I$ is set inclusion on the translations**

\[
\sigma \leq_I \sigma' \iff \{\sigma\} \supseteq \{\sigma'\}
\]
Simple types

\[ \alpha \rightarrow \alpha \]
Simple types

\[ \alpha \rightarrow \alpha \]
Simple types

\[ \alpha \rightarrow \alpha \]
System-F types

\[ \forall (\alpha) \, \alpha \rightarrow \alpha \]
System-F types

\[ \forall (\alpha) \alpha \rightarrow \alpha \]
System-F types

\[ \forall (\alpha) \forall (\beta) (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \beta \]
System-F types

\[ \forall(\alpha) \; \forall(\beta) \; (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \beta \]
**System-F types**

\[ \forall (\alpha) \forall (\beta) \ (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \beta \]

![Diagram of System-F types]
System-F types

\[(\alpha \to \beta) \to \alpha \to \beta\]

Sharing of inner nodes:
- Coming from the dag-representation of simple types.
- Canonical (unique) representation if disallowed.
**System-F types**

\( \forall (\alpha) \forall (\beta) (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \beta \)

\[ \leq F \]

\( \forall (\alpha) (\alpha \rightarrow \alpha) \rightarrow \alpha \rightarrow \alpha \)
**System-F types**

\[ \forall (\alpha) \forall (\beta) \ (\alpha \to \beta) \to \alpha \to \beta \]

\[ \forall (\alpha) \forall (\gamma) \ (\alpha \to \gamma \to \gamma) \to \alpha \to \gamma \to \gamma \]
System-F types

\[ \forall(\alpha) \ (\forall(\beta) (\alpha \rightarrow \beta)) \rightarrow \alpha \rightarrow \beta \]

\[ \forall(\alpha) (\alpha \rightarrow \forall(\gamma) \gamma \rightarrow \gamma) \rightarrow \alpha \rightarrow \forall(\gamma) \gamma \rightarrow \gamma \]
Types in $iML^F$

$\forall(\beta \geq \forall(\alpha) \alpha \to \alpha) \to \beta \to \beta$
Types in $iML^F$

$\forall (\beta \geq \forall (\alpha) \alpha \rightarrow \alpha) \rightarrow \beta \rightarrow \beta$
Types in \( i\text{ML}^F \)

\[
\forall (\beta \geq \forall (\alpha) \alpha \rightarrow \alpha) \rightarrow \beta \rightarrow \beta
\]
Types in $i\text{ML}^F$

$$\forall (\beta \geq \forall (\alpha) \alpha \rightarrow \alpha) \rightarrow \beta \rightarrow \beta$$
Types in $iML^F$

$$\forall(\beta \geq \forall(\alpha)\; \alpha \rightarrow \alpha) \rightarrow \beta \rightarrow \beta$$
Types in $iML^F$

$$\forall (\beta \geq \forall (\alpha) \alpha \rightarrow \alpha) \rightarrow \beta \rightarrow \beta$$

$$(\forall (\alpha) \alpha \rightarrow \alpha) \rightarrow \forall (\alpha) \alpha \rightarrow \alpha$$
Types in $iML^F$

$$\forall (\beta \geq \forall (\alpha) \alpha \rightarrow \alpha) \rightarrow \beta \rightarrow \beta$$

$$\forall (\alpha) (\alpha \rightarrow \alpha) \rightarrow (\alpha \rightarrow \alpha)$$
Types in $\text{iMLF}$

$$\forall(\beta \geq \forall(\alpha)\alpha \rightarrow \alpha) \rightarrow \beta \rightarrow \beta$$
Types in $i\text{ML}^F$

$$\forall (\beta \geq \forall (\alpha) \alpha \rightarrow \alpha) \rightarrow \beta \rightarrow \beta$$
Types in $i\text{MLF}^F$

$$\forall(\beta \geq \forall(\alpha) \alpha \to \alpha) \to \beta \to \beta$$
The semantics cannot be captured by
- a finite set of System-F types up to $\leq$
- a finite intersection type.
\( \forall (\beta \geq (\forall (\alpha) \alpha \rightarrow \alpha) \rightarrow (\forall (\alpha) \alpha \rightarrow \alpha)) \rightarrow \beta \rightarrow \beta \)
\( \forall (\beta \geq (\forall (\alpha) \alpha \rightarrow \alpha) \rightarrow (\forall (\alpha) \alpha \rightarrow \alpha)) \rightarrow \beta \rightarrow \beta \)
\( \forall (\beta \geq (\forall (\alpha) \alpha \rightarrow \alpha) \rightarrow (\forall (\alpha) \alpha \rightarrow \alpha)) \rightarrow \beta \rightarrow \beta \)
iMLF types

\( \forall (\beta \geq (\forall (\alpha) \alpha \rightarrow \alpha) \rightarrow (\forall (\alpha) \alpha \rightarrow \alpha)) \rightarrow \beta \rightarrow \beta \)
Type instance \( \leq \) in iMLF

Only four atomic instance operations, and only two new.
**Type instance \( \leq \) in \( iML^F \)**

*Only four atomic instance operations, and only two new.*

**Grafting**

![Diagram of grafting](image)
Type instance $\leq$ in $i\text{ML}^{\text{F}}$

Only four atomic instance operations, and only two new.

Grafting

Raising
Type instance $\leq$ in $i\text{ML}^F$

Only four atomic instance operations, and only two new.

- **Grafting**
- **Raising**
- **Merging**
Type instance $\leq$ in $iML^F$

Only four atomic instance operations, and only two new.

- **Grafting**
- **Raising**
- **Merging**
- **Weakening**
Checking the example choice id

Raising

Weakening

Didier Remy (INRIA-Rocquencourt)
Outline

1 Design
   - $i\text{ML}^F$: an implicitly-typed extension of System $F$
   - Types explained
   - $e\text{ML}^F$: an explicitly-typed version of $i\text{ML}^F$

2 Uses and Implementation
   - Examples
   - Type inference
   - Restrictions and extensions
Design of $eML^F$

Goal

Find a restriction $iML^F$ where programs that would require guessing polymorphism are ill-typed.

Guideline

Function parameters that are used polymorphically (and only those) need an annotation.
First-order inference with second-order types

Easy examples

\[ \lambda(z) \ z \quad : \quad \forall(\alpha) \ \alpha \rightarrow \alpha \quad \text{as in ML} \]

\[ \text{let } x = \lambda(z) \ z \ \text{in } x \ x \quad : \quad \forall(\alpha) \ \alpha \rightarrow \alpha \quad \text{as in ML} \]

\[ \lambda(x) \ x \ x \quad : \quad \text{ill-typed!} \quad x \ \text{is used polymorphically} \]

\[ \lambda(x : \forall(\alpha) \ \alpha \rightarrow \alpha) \ x \ x \quad : \quad (\forall(\alpha) \ \alpha \rightarrow \alpha) \rightarrow (\forall(\alpha) \ \alpha \rightarrow \alpha) \]
First-order inference of second order types

More challenging example

$$(\lambda(z)\ z)\ (a : \sigma)$$ where $\sigma$ is truly polymorphic
First-order inference of second order types

More challenging example

\((\lambda(z)\ z)\ (a : \sigma)\) where \(\sigma\) is truly polymorphic

- \(z\) must carry values of a polymorphic type.
- but \(z\) is not used polymorphically.
- Indeed, it can be typed in System F as

\[ (\Lambda \alpha. \ (\lambda(z : \alpha)\ z)\ [\sigma])\ (a : \sigma) \]
First-order inference of second order types

More challenging example

\[(\lambda(z)\ z)\ (a : \sigma)\] where \(\sigma\) is truly polymorphic

- \(z\) must carry values of a **polymorphic type**.
- but \(z\) is not used **polymorphically**.
- Indeed, it can be typed in System F as

\[(\Lambda\alpha.\ \lambda(z : \alpha)\ z)\ [\sigma] (a : \sigma)\]
First-order inference of second order types

More challenging example

$$\lambda(z) \ (z \ (a : \sigma))$$
First-order inference of second order types

More challenging example

\[ \lambda(z) (z (a : \sigma)) \]

- \( z \) have the polymorphic type \( \sigma \rightarrow \sigma \) ?
- \( z \) is node used polymorphically:
  polymorphism is only carried out from the argument to the result.
First-order inference of second order types

More challenging example

\[ \lambda(z) \, (z \, (a : \sigma)) \]

- \( z \) have the polymorphic type \( \sigma \rightarrow \sigma \)?
- \( z \) is node used polymorphically:
  polymorphism is only carried out from the argument to the result.
Abstracting second-order polymorphism as first-order types

Solution

1) Disallow second-order types under arrows, e.g. such as $\sigma_{id} \rightarrow \sigma_{id}$
2) Instead, allow type variables to stand for polymorphic types:

\[
\begin{align*}
\text{write} & \quad \forall (\alpha \Rightarrow \sigma_{id}) \quad \alpha \rightarrow \alpha \\
\text{read} & \quad “\alpha \rightarrow \alpha \text{ where } \alpha \text{ abstracts } \sigma_{id}” \\
\text{means} & \quad \sigma_{id} \rightarrow \sigma_{id}
\end{align*}
\]

Mechanism

1) function parameters must be monomorphic (but may be abstract).
2) forces all polymorphism to be abstracted away in the context.

\[
\begin{align*}
\alpha \Rightarrow \sigma_{id}, x : \alpha & \vdash x : \alpha \\
\alpha \Rightarrow \sigma_{id} & \vdash \lambda(x) \ x : \alpha \rightarrow \alpha \\
\lambda(x) \ x : \forall (\alpha \Rightarrow \sigma_{id}) \ & \alpha \rightarrow \alpha
\end{align*}
\]
Abstracting second-order polymorphism

Key point: abstraction is directional

\[ \alpha \Rightarrow \sigma \vdash \sigma \leq \alpha \]

Hence,

\[ \vdash a : \sigma \]

\[ \alpha \Rightarrow \sigma \vdash a : \alpha \quad \alpha \Rightarrow \sigma, \ z : \alpha \rightarrow \alpha \vdash z : \alpha \rightarrow \alpha \]

\[ \alpha \Rightarrow \sigma, \ z : \alpha \rightarrow \alpha \vdash z \ a : \alpha \]

\[ \alpha \Rightarrow \sigma \vdash \lambda(z) \ z \ a : (\alpha \rightarrow \alpha) \rightarrow \alpha \]

\[ \vdash \lambda(z) \ z \ a : \forall(\alpha \Rightarrow \sigma) \ (\alpha \rightarrow \alpha) \rightarrow \alpha \]
Abstracting second-order polymorphism

Key point: abstraction is directional

\[ \alpha \Rightarrow \sigma \vdash \sigma \leq \alpha \]

But,

\[ \alpha \Rightarrow \sigma_{id}, \ z : \alpha \vdash z : \alpha \]

\[ \alpha \Rightarrow \sigma_{id}, \ z : \alpha \vdash z : \sigma_{id} \]

\[ \alpha \Rightarrow \sigma_{id}, \ z : \alpha \vdash z : \alpha \rightarrow \alpha \]

\[ \alpha \Rightarrow \sigma_{id}, \ z : \alpha \vdash \lambda(z) \ z z : \alpha \rightarrow \alpha \]

\[ \vdash \lambda(z) \ z z : \forall(\alpha \geq \sigma_{id}) \ \alpha \rightarrow \alpha \]
Types in eMLF

Introduce a new binder for abstraction

\[ \forall (\alpha \Rightarrow \forall (\beta) \beta \rightarrow \beta) \alpha \rightarrow \alpha \]
Types in eMLF

Introduce a new binder for abstraction

$$\forall (\alpha \Rightarrow \forall (\beta) \beta \to \beta) \alpha \to \alpha$$
Types in eMLF

Introduce a new binder for abstraction

\[ \forall (\alpha \Rightarrow \forall (\beta) \, \beta \rightarrow \beta) \, \forall (\alpha' \Rightarrow \forall (\beta) \, \beta \rightarrow \beta) \, \alpha \rightarrow \alpha' \]
Types in $eML^F$

Introduce a new binder for abstraction

$$\forall (\alpha \Rightarrow \forall (\beta) \beta \rightarrow \beta) \forall (\alpha' \geq \forall (\beta) \beta \rightarrow \beta) \alpha \rightarrow \alpha'$$
Types, graphically

= first-order term-dag + a binding tree
Types, graphically

= first-order term-dag + a binding tree
Types, graphically

= first-order term-dag + a binding tree
Types, graphically

= first-order term-dag + a binding tree

+ well-formedness conditions relating the two
Type instance $\leq$ in eML$^F$

Sharing and binding of abstract nodes matter

Grafting, Merging, Raising, Weakening
Unchanged.
Type annotations

Recovering the missing power

\((\leq) \subset (\leq_i)\)

- \(\leq\) is weaker than \(\leq_i\), as sharing and binding of abstract nodes matters.
Type annotations

Recovering the missing power

$\leq \subseteq (\leq_I) = (\leq \cup \diamond I)^* = (\leq ; \diamond I)$

- $\leq$ is weaker than $\leq_I$, as sharing and binding of abstract nodes matters.
- Use **explicit type annotations** to recover $(\diamond I \setminus \leq)$.
  Notice that the weaker $\leq$, the more annotations will be required.
Type annotations

Recovering the missing power

\[(\leq) \subset (\leq_i) = (\leq \cup \diamond i)^* = (\leq; \diamond i)\]

- Intuitively,
  \[
  \Gamma \vdash a : \tau \quad \tau \leq i \tau' \quad \prod
  \Gamma \vdash (a : \tau') : \tau'
  \]

- Actually, use coercion functions:
  \[
  (- : \sigma) : \quad \forall (\alpha \Rightarrow \sigma) \quad \forall (\alpha' \Rightarrow \sigma) \quad \alpha \rightarrow \alpha'
  \]

- Add syntactic sugar \(\lambda(x : \sigma) a\)
  \[
  \triangleq \lambda(x) \text{ let } x = (x : \sigma) \text{ in } a
  \]
  \[
  \equiv \lambda(x) \ a[x \leftarrow (x : \sigma)]
  \]
Type annotations

Recovering the missing power

\[(\leq) \cap (\leq_I) = (\leq \cup \Diamond_I)^* = (\leq; \Diamond_I)\]

- Intuitively,

\[\frac{\Gamma \vdash a : \tau \quad \tau \subseteq_I \tau'}{\Gamma \vdash (a : \tau') : \tau'}\]

- Actually, use coercion functions:

\[(_ : \exists (\beta) \sigma) : \forall(\beta) \forall(\alpha \Rightarrow \sigma) \forall(\alpha' \Rightarrow \sigma) \alpha \rightarrow \alpha'\]

- Add syntactic sugar \[\lambda(x : \sigma) a \triangleq \lambda(x) \text{ let } x = (x : \sigma) \text{ in } a\]

\[\equiv \lambda(x) a[x \leftarrow (x : \sigma)]\]
Type annotations

Remember \( \alpha \Rightarrow \sigma, x : \alpha \vdash x : \sigma \)

- Prevents typing \( \lambda(x) \ x \ x \)

With an annotation \( \alpha \Rightarrow \sigma, x : \alpha \vdash (x : \sigma) : \sigma \)

- Allows typing \( \lambda(x : \sigma_{id}) \ x \ x \)
Outline

1 Design
   - $i\text{ML}^F$: an implicitly-typed extension of System F
   - Types explained
   - $e\text{ML}^F$: an explicitly-typed version of $i\text{ML}^F$

2 Uses and Implementation
   - Examples
   - Type inference
   - Restrictions and extensions
About principal types

Fact

- Programs have principal types, given with their type annotations.

Programs with type annotations

- Two versions of the same program, but with different type annotations, usually have different principal types.

Programs typable without type annotations

- Exactly ML programs.
- But usually have a more general type than in ML (*e.g.* choice id)
- Annotations may still be useful to get more polymorphism.
Robustness to small program transformations

Agreed

- Programmers must be free of choosing their programming patterns/styles.
- Programs should be maintainable.

Therefore

- Programs should be stable under some small, but important program transformations.
Robustness to small program transformations

\( a \subseteq a' \) means all typings of \( a \) are typings of \( a' \)

Let-conversion

\[ \text{let } x = a_1 \text{ in } a_2 \equiv a_2[x \leftarrow a_1] \]

Common subexpression can be factored out.

Redefine application

\[ a_1 \ a_2 \equiv (\lambda(f) \ \lambda(x) \ f \ x) \ a_1 \ a_2 \]

Many functionals, such as maps are typed as applications.

\( \eta \)-conversion of functional expressions

\[ a \equiv \lambda(x) \ a \ x \]

Delay the evaluation.

Reordering of arguments

\[ a \ a_1 \ a_2 \equiv (\lambda(x) \ \lambda(y) \ a \ y \ x) \ a_2 \ a_1 \]

Curryfication

\[ a \ (a_1, a_2) \equiv (\lambda(x) \ \lambda(y) \ a \ (x, y)) \ a_1 \ a_2 \]

All valid in \( \text{MLF} \)
Printing types

Only overlined bindings need to be drawn

Leave implicit bindings that are

- at unshared, inner nodes,
- bound just above,
- abstractions on the left of arrows,
- instances on the right arrows.

\[
(\forall (\alpha) \forall (\beta) (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \beta)) \rightarrow (\forall (\alpha) \alpha \rightarrow \alpha) \rightarrow (\forall (\alpha) \alpha \rightarrow \alpha)
\]
Printing types

**Only overlined bindings need to be drawn**

- at unshared, inner nodes,
- bound just above,
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- instances on the right arrows.

\[
(\forall (\alpha) \forall (\beta) (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \beta)) \rightarrow \forall (\gamma \Rightarrow \forall (\alpha) \alpha \rightarrow \alpha) (\forall (\alpha) \alpha \rightarrow \alpha) \rightarrow \gamma
\]
Printing types

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Leave implicit bindings that are

- at unshared, inner nodes,
- bound just above,
- abstractions on the left of arrows,
- instances on the right arrows.

\[ \forall (\gamma \Rightarrow \forall (\alpha) \; \alpha \rightarrow \alpha) \; (\forall (\alpha) \; \forall (\beta) \; (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \beta)) \rightarrow (\forall (\alpha) \; \alpha \rightarrow \alpha) \rightarrow \gamma \]
Printing types

Only overlined bindings need to be drawn

Leave implicit bindings that are

- at unshared, inner nodes,
- bound just above,
- abstractions on the left of arrows,
- instances on the right arrows.

\[(\forall (\alpha) \forall (\beta) (\alpha \to \beta) \to (\alpha \to \beta)) \to \forall (\gamma \geq \sigma_{id}) \gamma \to \gamma\]
Examples

Library functions

```ocaml
let rec fold f v = function
    | Nil -> v
    | Cons (h, t) -> fold f (f h t) t;;
```

```
val fold : \(\forall (\alpha) \forall (\beta) (\alpha \rightarrow \alpha \ list \rightarrow \beta) \rightarrow \beta \rightarrow \alpha \ list \rightarrow \beta
```

Few type annotations are needed in practice

- No dummy/annoying/unpredictable annotations.

Output types are usually readable

- Most inner binding edges may be left implicit.
- Many library functions libraries keep their ML type in ML\(^F\), modulo the syntactic sugar.
More examples

Church’s numerals

\begin{align*}
\textbf{type} \; \texttt{nat} &= \forall (\alpha) \; (\alpha \to \alpha) \to \alpha \to \alpha ;; \\
\textbf{let} \; \texttt{zero} &= \textbf{fun} \; f \; x \to x ;; \\
\textbf{val} \; \texttt{zero} &: \forall (\alpha) \; \alpha \to (\forall (\beta) \; \beta \to \beta)
\end{align*}

With type annotations on the iterator

\begin{align*}
\textbf{let} \; \texttt{succ} \; (n : \texttt{nat}) &= \textbf{fun} \; f \; x \to n \; f \; (f \; x) ;; \\
\textbf{val} \; \texttt{succ} &: \texttt{nat} \to (\forall (\alpha) \; (\alpha \to \alpha) \to \alpha \to \alpha) \\
\textbf{let} \; \texttt{add} \; (n : \texttt{nat}) \; m &= n \; \texttt{succ} \; m ;; \\
\textbf{val} \; \texttt{add} &: \texttt{nat} \to (\forall (\alpha) \; (\alpha \to \alpha) \to \alpha \to \alpha) \\
\textbf{let} \; \texttt{mul} \; n \; (m : \texttt{nat}) &= m \; (\texttt{add} \; n) \; \texttt{zero} ;; \\
\texttt{mul} &: \texttt{nat} \to \texttt{nat} \to (\forall (\alpha) \; (\alpha \to \alpha) \to \alpha \to \alpha)
\end{align*}
More examples

Church’s numerals

type nat = ∀ (α) (α → α) → α → α;;
let zero = fun f x → x;;
val zero : ∀(α) α → (∀(β) β → β)

Without type annotations

let succ n = fun f x → n f (f x);;
val succ : ∀ (α, β, γ) ((α → β) → β → γ) → (α → β) → α → γ
let add n m = n succ m;;
val add : ∀(δ ≥ ∀(α,β,γ,ε,φ) ((α → β) → β → γ) → (α → β) → α → γ)
        ∀(ε,φ) (δ → ε → φ) → ε → φ

In ML:

val add : ∀ (α,β,γ,ε,φ) (((((α → β) → β → γ) → (α → β) → α → γ)
                           → ε → φ) → ε → φ
More examples

Church’s numerals

define the type nat and the values zero and 1

```ml
let zero = fun f x -> x

val zero : \alpha. \alpha -> (\alpha -> \alpha)
```

```ml
let mul n m = m (add n) zero

let mul' = (mul : nat -> nat -> nat)
```

This works because MLF infers the types

```ml
let mul n m = m (add n) zero

let mul' = (mul : nat -> nat -> nat)
```

MLF without any type annotation at all does not do better than ML!
Unification algorithm

Computes principal unifiers, in three steps

- Computes the underlying dag-structure by first-order unification.
- Computes the binding structure
  - by raising binding edges
  - as little as possible to maintain well-formedness.
- Checks that no locked binding edge (in red) has been raised or merged.

Complexity

- Same as first-order unification. Other passes are in linear time.
- $O(n)$ (or $O(n\alpha(n))$ if incremental).

Note

- The algorithm performs “first-order unification of second-order types”.

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Type inference

Proceeds much as in ML

- Implement type-instantiation by copying the polymorphic part.
- Use unification to solve typing constraints.
- Generalize as much as possible at every step (not just at every let).

Type inference with typing constraints

**Complexity in** $O(kn(\alpha(kn) + d)) \approx O(kdn)$

- As for ML (see McAllester).
  - $k$ is the maximal size of types (usually not too large)
  - $d$ is the maximal nesting of type schemes.
- However, ML and MLF differs on $d$, which is:
  - the left-nesting of let-bindings in ML
  - the maximum height of an expression in MLF
    (Still, does not grow on the right of let-bindings).
Variations on $\text{ML}^F$

**Shallow $\text{ML}^F$**

The version we presented is a “downgraded” version of $\text{ML}^F$.

- Types are stratified.
- Instance bounded types cannot appear in bounds of abstract variables.
- In particular, type annotations must be $F$ types.
Variations on $\text{MLF}$

**Shallow $\text{MLF}$**

The version we presented is a “downgraded” version of $\text{MLF}$.

- Types are stratified.
- Instance bounded types cannot appear in bounds of abstract variables.
- In particular, type annotations must be F types.

**Full $\text{MLF}$**

- No stratification, more expressive.
- All interesting properties are preserved.
- Algorithms are mostly unchanged.
- We lose the interpretation of types as sets of System-F types.
Variations on $\text{MLF}^F$

**Shallow $\text{MLF}^F$**
The version we presented is a “downgraded” version of $\text{MLF}^F$.

- Types are stratified.
- Instance bounded types cannot appear in bounds of abstract variables.
- In particular, type annotations must be $F$ types.

**Simple $\text{MLF}^F$**
Remove instance bindings $\geq$, keep abstract bindings $\Rightarrow$.

- Equivalent to System F.
- Principal types are lost (no type inference).
A hierarchy of languages

- (Full) ML$^F$
- Shallow ML$^F$
- Simple ML$^F$
- Simple Types

ML

ML$^F$

F
A hierarchy of languages
A hierarchy of languages

(Full) ML^F

Shallow ML^F

Rigid ML^F

F

ML

Simple Types

Simple ML^F

Rigid ML^F
Extensions

Primitive Existential types

- Encoding with existential types works well (only annotate at creation).
- Can more be done with primitive existential?

(Equi-) recursive types

- Easy when cycles do not contain quantifiers.
- Cycles that crosses quantifiers are difficult.

Higher-order types

- Use two quantifiers (explicit coercions between the two permitted)
  - $\forall^F$ for fully explicit type abstractions and
  - $\forall^{ML^F}$ for implicit $ML^F$ polymorphism.
- Restrict $\forall^{ML^F}$ to the first-order type variables.
- Can $\forall^{ML^F}$ also be used at higher-order kinds?
Papers and prototypes

Talk mainly based on
- Recasting-MLF with Didier Le Boltan.
- A Graphical Presentation of MLF Types, with Boris Yakobowski.

Other papers and online prototype at
- http://gallium.inria.fr/~remy/mlf/

See also Daan Leijen’s papers and prototypes
- http://research.microsoft.com/users/daan/pubs.html
Conclusions

Just two things to remember

- MLF allows function parameters to implicitly carry polymorphic values that are used monomorphically.
- Type annotations are required only to allow function parameters to carry (polymorphic) values that are used polymorphically.

MLF design, use, and implementation are close to ML

- MLF piggy-backs on ML type-shemes and generalization mechanism.
- Part of the credits should be returned to the great designers of ML.

Hopefully

- ML users will feel “at home”.
- Other users will also appreciate the convenience of type inference.
Appendix

3 Type inference demo

4 More examples: encoding of existential types

5 About Rigid ML$^F$

6 Questions
   - What is an Intermediate language for ML$^F$
   - Sharing of abstract nodes is irreversible (implicitly)

7 Details of slides
   - Another example of System F types
   - Abstraction in action
Type inference with typing constraints (demo)

\[ \lambda(x) \ x \]
Type inference with typing constraints (demo)

\[ \lambda(x) \ x \]
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Type inference with typing constraints (demo)
Type inference with typing constraints (demo)

\[
\text{let } y = \lambda(x) \ x \\
\text{in } y \ y
\]
Type inference with typing constraints (demo)

```ml
let y = λ(x) x
   in y y
```

```
λ(x) x
```

```
y y
```

```
y
```

```
```

```
```

Type inference with typing constraints (demo)
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\[ \lambda(z) \ z \ (\lambda(x) \ x) \]
Type inference with typing constraints (demo)

\[ \lambda(z)\, z\, (\lambda(x)\, x) \]
Type inference with typing constraints (demo)
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Type inference with typing constraints (demo)
Type inference with typing constraints (demo)
Type inference with typing constraints (demo)
Type inference with typing constraints (demo)

\[ \lambda(z) \ (z : \sigma_{id}) \]
Type inference with typing constraints (demo)

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back
More examples

Encoding of existential types, e.g. $\exists \beta. \beta \times \beta \to \alpha$

\[
\text{type } \alpha \ \text{func} = \forall (\gamma) \ \forall (\delta = \forall (\beta) \ \beta \times (\beta \to \alpha) \to \gamma) \ \delta \to \gamma
\]

\[
\text{val } \text{pack } z = \text{fun } (f : \exists (\gamma) \ \forall (\beta) \ \beta \times (\beta \to \alpha) \to \gamma) \to f z;;
\]

\[
\text{val } \text{pack : } \forall (\alpha) \ \forall (\beta) \ \alpha \times (\alpha \to \beta) \to (\forall (\gamma) \ \forall (\delta) \ \delta \times (\delta \to \beta) \to \gamma) \to \gamma)
\]

\[
\text{let } \text{packed_int } = \text{pack } (1, \text{fun } x \to x+1);;
\]

\[
\text{let } \text{packed_pair } = \text{pack } (1, \text{fun } x \to (x, x));;
\]

\[
\text{let } v = \text{packed_int } (\text{fun } p \to (\text{snd } p) \ (\text{fst } p));;
\]
About Rigid $\text{ML}^F$

**Rigid $\text{ML}^F$ lies very close to $\text{ML}^F$**

- It uses and relies on (Shallow) $\text{ML}^F$ internally.
- It projects $\text{ML}^F$ principal types into System-F types at let-bindings, by raising variable bindings as much as possible.

**Rigid $\text{ML}^F$ looses important properties of $\text{ML}^F$**

- There are no principal types *per se*.
  - Rigid $\text{ML}^F$ pretends to have principal types, but this is in an ad hoc manner, using a non logical typing rule for Let-bindings with a premise that blocks free uses of type-instantiation.
- $\text{let } x = \lambda(z : \sigma) z \text{ in } a_2$ may be accepted while $\text{let } x = \lambda(z) z \text{ in } a_2$ would be rejected.
- Rigid $\text{ML}^F$ is not invariant by let-expansion (which signs the lost of truly principal types).
About Rigid $\text{MLF}$

Rigid $\text{MLF}$ lies very close to $\text{MLF}$

- It uses and relies on (Shallow) $\text{MLF}$ internally.
- It projects $\text{MLF}$ principal types into System-F types at let-bindings, by raising variable bindings as much as possible.

Rigid $\text{MLF}$ looses important properties of $\text{MLF}$

- There are no principal types per se.
- Rigid $\text{MLF}$ is not invariant by let-expansion (which signs the lost of truly principal types).

Rigid $\text{MLF}$ is a subset of System F

- This is both its interest and its problem.
What would be an intermediate language for $ML^F$?

Problem

- Subject reduction is only proved in $iML^F$, which has the same type erasure as $eML^F$.
  - This ensures correctness of $iML^F$
  - But does not help to propagate annotations during reduction (or other program transformations)
- Even so, $eML^F$ requires type inference, which is not a local process.

Solution

- Introduce a fully explicit version of $xML^F$ (easy)
- Instrument reduction rules to keep track of types during reduction (not entirely trivial)
- This has to be investigated.
Sharing of abstract nodes is irreversible (implicitly)

Can you show an example illustrating the difference?

Fact: $\forall (\alpha \Rightarrow \sigma) \, \alpha \to \alpha \nleq \forall (\alpha \Rightarrow \sigma, \alpha' \Rightarrow \sigma) \, \alpha \to \alpha'$

Observe that:

- $\lambda(z) \, z : \forall (\alpha \Rightarrow \sigma) \, \alpha \to \alpha$
- $(\_ : \sigma) : \forall (\alpha \Rightarrow \sigma, \alpha' \Rightarrow \sigma) \, \alpha \to \alpha'$

Then, the context $a \triangleright \lambda(\times) \left[\right] \times \times$ distinguishes those two expressions.

- $a[\lambda(z) \, z]$ is ill-typed.
  (As it uses no type annotation and it is ill-typed in ML)
- $a[(\_ : \sigma)]$ is well-typed.
System-F types (encoding of existential types)

\[ \forall (\alpha) \ (\forall (\beta) \ \tau_\beta \rightarrow \alpha) \rightarrow \alpha \]
**System-F types (encoding of existential types)**

\[ \forall (\alpha) \ (\forall (\beta) \ \tau_\beta \rightarrow \alpha) \rightarrow \alpha \]

\[ (\forall (\beta) \ \tau_\beta \rightarrow \forall (\alpha) \ \alpha \rightarrow \alpha) \rightarrow \forall (\alpha) \ \alpha \rightarrow \alpha \]
System-F types (encoding of existential types)

\[ \forall (\alpha) \ (\forall (\beta) \ \tau_\beta \rightarrow \alpha) \rightarrow \alpha \]
Type annotations

\[ \alpha \Rightarrow \sigma, \beta \Rightarrow \sigma \vdash \sigma \leq \alpha \text{ and } \sigma \leq \beta \]
\[ \alpha \Rightarrow \sigma, \beta \Rightarrow \sigma \vdash \forall (\alpha' \Rightarrow \sigma) \forall (\beta' \Rightarrow \sigma) \alpha' \rightarrow \beta' \leq \forall (\alpha' \Rightarrow \alpha) \forall (\beta' \Rightarrow \beta) \alpha' \rightarrow \beta' \leq \alpha \rightarrow \beta \]

\[ \alpha \Rightarrow \sigma, x : \alpha, \beta \Rightarrow \sigma \vdash (\_ : \sigma) : \alpha \rightarrow \beta \]
\[ \alpha \Rightarrow \sigma, x : \alpha, \beta \Rightarrow \sigma \vdash x : \alpha \]
\[ \alpha \Rightarrow \sigma, x : \alpha, \beta \Rightarrow \sigma \vdash (x : \sigma) : \beta \]
\[ \alpha \Rightarrow \sigma, x : \alpha \vdash (x : \sigma) : \forall (\beta \Rightarrow \sigma) \beta \]
\[ \alpha \Rightarrow \sigma, x : \alpha \vdash (x : \sigma) : \sigma \]
Type annotations

\[ \alpha \Rightarrow \sigma_{id}, x : \alpha \vdash (x : \sigma_{id}) : \sigma_{id} \]

\[ \alpha \Rightarrow \sigma_{id}, x : \alpha \vdash (x : \sigma_{id}) : \alpha \rightarrow \alpha \quad \alpha \Rightarrow \sigma_{id}, x : \alpha \vdash x : \alpha \]

\[ \alpha \Rightarrow \sigma_{id}, x : \alpha \vdash \lambda(x) \ (x : \sigma_{id}) \ x : \alpha \rightarrow \alpha \]

\[ \vdash \lambda(x) \ (x : \sigma_{id}) \ x : \forall(\alpha \Rightarrow \sigma_{id}) \ \alpha \rightarrow \alpha \]