Abstract

Over the past two decades, many concurrent data structures have been designed and implemented. Nearly all such work analyzes concurrent data structures empirically, omitting asymptotic bounds on their efficiency, partly because of the complexity of the analysis needed, and partly because of the difficulty of obtaining relevant asymptotic bounds: when the analysis takes into account important practical factors, such as contention, it is difficult or even impossible to prove desirable bounds.

In this paper, we show that considering structured concurrency or relaxed concurrency models can enable establishing strong bounds, also for contention. To this end, we first present a dynamic relaxed counter data structure that indicates the non-zero status of the counter. Our data structure extends a recently proposed data structure, called SNZI, allowing our structure to grow dynamically in response to the increasing degree of concurrency in the system.

Using the dynamic SNZI data structure, we then present a concurrent data structure for series-parallel directed acyclic graphs (sp-dags), a key data structure widely used in the implementation of modern parallel programming languages. The key component of sp-dags is an in-counter data structure that is an instance of our dynamic SNZI. We analyze the efficiency of our concurrent sp-dags and in-counter data structures under nested-parallel computing paradigm. This paradigm offers a structured model for concurrency. Under this model, we prove that our data structures require amortized \( O(1) \) shared memory steps, including contention. We present an implementation and an experimental evaluation that suggests that the sp-dags data structure is practical and can perform well in practice.

1. Introduction

Non-blocking data structures are commonly used in practical concurrent systems because such data structures allow multiple processes to access the structure concurrently while guaranteeing that the whole system will make progress. As discussed by the survey of Moir and Shavit [38], many non-blocking data structures have been designed and implemented for various problems, such as shared counters [22, 42], non-zero indicator counters [14], stacks [45], queues [37], doubly ended queues [7], and binary search trees [9, 13].

The theoretical efficiency of non-blocking data structures, however, has received relatively little attention. One reason for this lack of attention is that in a non-blocking data structure, some processes may get stuck, leading to seemingly infinite execution times. Another reason is that the common shared-memory models, under which concurrent data structures are analyzed, usually do not account for an important practical concern: the effects of memory contention. These effects can be significant especially as the degree of concurrency increases [12, 38, 43].

We are interested in the design and analysis of provably efficient non-blocking data structures in realistic models of concurrency that account for the cost of contention. This goal represents a major challenge, because contention often factors into the running time as a linear additive factor (e.g., [13, 16, 18, 39]). A series of papers have also established linear-time lower bounds [16, 27, 30] when contention is taken into account by using models that allow only one process to access a memory object [5, 11, 12]. For example, Fich, Hendler, and Shavit [16] show that, when accounting for contention, a number of concurrent data structures, including counters, stacks, and queues, require \( \Omega(n) \) time, where \( n \) is the number of processes/threads in the system, a.k.a., the degree of concurrency.

In this paper, we show that a relaxation of the general concurrency model can unlock the design and analysis of provably efficient concurrent data structures, accounting also for contention. As a relaxation, we consider a structured concurrent programming paradigm, nested parallelism, where concurrency is created under the control of programming primitives, such as fork-join and async-finish. Nested parallelism
is broadly available in a number of modern programming languages and language extensions, such as OpenMP, Cilk [19], Fork/Join Java [34], Habanero Java [28], TPL [35], TBB [29], X10 [10], parallel ML [17], and parallel Haskell [32]. In these systems, nested-parallel programs can be expressed by using a primitives such as fork-join (a.k.a., spawn-sync), and async-finish. When executed, a nested parallel program can create many threads, leading to a high degree of concurrency, but in a structured fashion: threads may be created by other threads via a async (or fork) operation and can also terminate and synchronize at known join (or finish) points. The fork-join primitives allow only two threads to synchronize at a join. In contrast, with async-finish primitives, any number of threads can terminate and synchronize at a single finish point. This structured nature of thread creation, termination, and synchronization is the key difference between nested parallelism and general concurrency.

Our starting point is prior work on relaxed concurrent counters, also called indicators, that indicate whether a counter is positive or not. Indicators contrast with exact counters as, for example, those considered by Herlihy, Shavit, and Waarts [27]. In prior work, Ellen et al. [14] present the SNZI (read “snazzy”), short for Scalable Non-Zero Indicator, data structure, prove it to be linearizable, and evaluate it empirically. They do not, however, provide any analytical upper bounds. Indeed, SNZI data structure could experience contention as multiple operations climb up the SNZI tree. In this paper, we present an extension to the SNZI data structure to allow it to grow dynamically at run time, perhaps in response to increasing degree of concurrency and thus increasing potential for contention (Section 2).

We then consider an application of our dynamic SNZI data structure to nested parallel computations, specifically for representing series-parallel directed acyclic graphs or sp-dags with unbounded in-degree vertices, which arise for example with parallel loops and async-finish primitives. Variants of dag data structures are used broadly in the implementation of modern parallel programming systems [10, 17, 19, 28, 29, 32, 34, 35]. These languages and systems typically represent the computation as a dag where each vertex corresponds to a fine-grained thread and schedule the vertices of the dag over the processors to minimize completion time. For efficiency reasons, it is important for the dag data structure to be non-blocking and low-contention, because a nested parallel program can create a dynamically varying set of fine-grained threads, consisting of anywhere from several threads to millions or more, depending on the input size.

The crux of the sp-dags problem is determining when a vertex in the dag becomes ready, i.e., all of its dependencies have been executed. Such readiness detection requires a concurrent dependency counter, which we call in-counter, that counts the number of incoming dependencies for each thread. When a dependency between a thread u and v is created, the in-counter of v is incremented; when u terminates its in-counter is decremented; when its in-counter reaches zero, vertex v becomes ready and can be executed. We show that in-counters can be implemented efficiently by using our dynamic SNZI data structure (Section 3). We prove that by managing carefully the growth of our dynamic SNZI data structure, we can guarantee that all operations require amortized O(1) time, including also contention (Section 4).

Finally, we present an implementation and perform an experimental evaluation in comparison to prior work (Section 5). Our results offer empirical evidence for the practicality of our proposed techniques, showing that our approach can perform well in practice.

Placed in the context of previously proven lower bounds for non-blocking data structures, the key to our result is the structured concurrency model that we consider. Our contributions are as follows.

- Dynamic SNZI: an extension to the original SNZI data structure that allows it to grow varying numbers of threads.
- Non-blocking low-contention sp-dags: a non-blocking data structure for series-parallel dags that can be used to represent nested-parallel computations provably efficiently and with low contention.
- Analysis: proofs that our series-parallel dag data structure experiences low contention under a nested parallel programming model.
- Evaluation: implementation and empirical evaluation that shows that our data structure is practical and that it can perform well in practice.

Due to space restrictions, we present some proof details and more experiments in the full version of the paper, which is available as a technical report [1].

1.1 Model

We consider a standard model of asynchronous shared memory where threads can communicate through taking atomic read-modify-write steps such as read, write and compare-and-swap (CAS). Threads can use these primitives to implement high-level operations. We define the contention experienced by an operation as the maximum number of non-trivial, shared memory steps that concurrently access the same memory location over all executions. A non-trivial step is one that might change the value of the memory location to which it is applied, and a non-trivial operation is an operation that takes at least one non-trivial step.

Our definition of contention is similar to stalls as defined by Fich et al. in [16]. Their definition, inspired by Dwork et al.’s work on contention models [12], considers a model in which each non-trivial memory step on a memory location must operate in isolation, and the rest of the operations stall.

For our bounds, we consider computations that are structured in a nested-parallel fashion, where threads execute independently in parallel or serially and synchronize at their
termination points. A series-parallel computation starts with a single root thread and constructs, as it executes, a series-parallel directed acyclic graph (sp-dag) of vertices where each vertex represents a thread of control, and edges represent dependencies between threads.

2. Dynamic SNZI

Concurrent counters are an important data structure used in many algorithms but can be challenging to implement efficiently when accounting for contention, as shown by the linear (in the degree of concurrency) lower bound of Fich et al [16]. Observing that the full strength of a counter is unnecessary in many applications, Ellen et al proposed non-zero indicators as relaxed counters that allow querying not the exact value of the counter but its sign or non-zero status [14]. Their non-zero indicator data structure, called SNZI, achieves low-contention by using a tree of SNZI nodes, each of which can be used to increment or decrement the counter. Determining the non-zero status of the counter only requires accessing the root of the tree. The tree is updated by increment and decrement operations. These operations are filtered on the way up to the root so that few updates reach the root.

The SNZI data structure is flexible. Each SNZI node interacts with its parent and its children, and will retain its correctness regardless of how many children it has. Thus, a SNZI tree can have any shape but it can be difficult to know the optimal shape of the tree a priori. In this section, we present a simple extension to SNZI that allows the tree to be grown dynamically, and show that our extension preserves correctness. In Section 3, we demonstrate how to use dynamic SNZI in a specific application—namely, to keep track of dependencies in an sp-dag efficiently.

Figure 1 illustrates the interface for the SNZI data structure and pseudo-code for the SNZI node struct snzi_node, the basic building block of the data structure from [14]. A SNZI node contains a counter c, a version number v, an array of children (two in our example), and a parent. The value at the counter indicates the surplus of arrivals with respect to the departures. The arrive and depart operations increment and decrement the value of the counter at the specified SNZI node a respectively, and query, which must be called at the root of the tree, indicates whether the number of arrive operations in the whole tree is greater than number of depart operations since creation of the tree.

To ensure efficiency and low contention, the SNZI implementation carefully controls the propagation of an arrival or departure at a node up the tree. The basic idea is to propagate a change to a node’s parent only if the surplus at that node flips from zero to positive or vice versa. More specifically, an arrive at node a is called on a’s parent if and only if a had surplus 0 at the beginning of the operation. Similarly, a depart at a node a is recursively called on a’s parent if and

```c
type snzi_node =
struct {
  (+ counter *)
  c:N∪{1}; initially 0
  (+ version number *)
  v:N; initially 0
  children : array of 2 snzi nodes; initially [null, null]
  parent : snzi_node
}

void fun snzi_arrive (snzi_node a) = ...
void fun snzi_depart (snzi_node a) = ...
bool fun snzi_query (snzi_node a) = ...
```

Figure 1: Partial pseudocode for the SNZI data structure; full description can be found in original SNZI paper [14].
which dictates how likely it is to create new children for the
array to be returned. This guarantees that any adversary that
before the coin flip happen
them to return no children. This property is ensured by having
even if all calls to
in expectation, regardless of the timing of the calls. That is,
probabilistic growing allows an application to control the rate
p
a coin toss with probability
for some of the threads using it. Such probabilistic growing is
probabilistic growing is useful when one wants a balance between low contention and
frequent memory allocation. The right probability threshold
to use, however, may depend not only on the application, but
also on the system.

If node \( u \) does not have any children at the end of a \texttt{grow}
operation, we have the operation return two pointers to \( u \) itself.
This return value is convenient for the in-counter application
that we present in the rest of the paper. Other operations
using dynamic SNZI may return anything else in that case.

The \texttt{grow} operation may be called at any time on any SNZI
node. To keep the flexible structure of the SNZI data structure,
we do not specify when to use the \texttt{grow} operation on the tree,
instead leaving it to the application to specify how to do so.
For example, in Section 3.3, we show how to use the \texttt{grow}
operation to implement a low-contention dependency counter
for dags. It is easy to see that the operation does not break
the linearizability of the SNZI structure; it is completely
independent from the count, version number, and parent
fields of nodes already in the tree, which are the only fields
that affect the correctness and linearizability.

We note a key property of the \texttt{grow} operation. We would
like to ensure that, given a probability \( p \), when \texttt{grow} is called
on a childless node, only \( 1/p \) such calls will return no children
in expectation, regardless of the timing of the calls. That is,
even if all calls to \texttt{grow} are concurrent, we want only \( 1/p \) of
them to return no children. This property is ensured by having
the coin flip happen before reading the value of the \textit{children}
array to be returned. This guarantees that any adversary that
does not know the result of local coin flips cannot cause more
than \( 1/p \) calls to return no children in expectation.

Dynamically shrinking the SNZI tree is more difficult,
because we need to be sure that no operation could access a
deallocated node. In general, one may easily and safely delete
a SNZI node from the tree if the following two conditions
hold: (1) its surplus is 0, and (2) no thread other than the one
deleting the node can reach it. The second condition is much
trickier to ensure. In Section 4 we show how to deallocate
nodes in our dependency counter data structure.

3. Series-Parallel Dags

A nested-parallel program can be represented as a series-
parallel dag, or an sp-dag, of threads, where vertices are
pieces of computation, and edges represent dependencies
between them. In fact, to execute such a program, modern
programming languages construct an sp-dag and schedule it
on parallel hardware. We present a provably low-contention
sp-dag data structure that can be used to execute nested-
parallel programs. After defining sp-dags, we present in
Section 3.1 a data structure for sp-dags by assuming an
in-counter data structure that enables keeping track of the
dependencies of a dag vertex (thread). We then present our
data structure for in-counters in Section 3.3.

A serial-parallel dag, or an sp-dag for short, is a directed-
acyclic graph (dag) that has a unique source vertex, which
has indegree 0, and a unique terminal vertex, which has
out-degree 0 and that can be defined iteratively. In the base
case, an sp-dag consists of a source vertex \( s \), a terminal
vertex \( v \), and the edge \((u,v)\). In the iterative case, an
sp-dag \( G = (V,E)\) with source \( s \) and terminal \( t \) is defined by
serial or parallel composition of two disjoint sp-dags \( G_1 \) and
\( G_2 \), where \( G_1 = (V_1,E_1)\) with source \( s_1 \) and terminal \( t_1 \) and
\( G_2 = (V_2,E_2)\) with source \( s_2 \) and terminal \( t_2 \), and \( V_1 \cap V_2 = \emptyset\):

- **serial composition**: \( s = s_1, t = t_2, V = V_1 \cup V_2, E = E_1 \cup E_2 \cup \{(1,s_2), (s_1,t_1)\} \);
- **parallel composition**: \( s, t \not\in (V_1 \cup V_2), V = V_1 \cup V_2 \cup \{(s,t), (s_1,t_1), (t_2,s_2)\} \),
and \( E = E_1 \cup E_2 \cup \{(s,s_2), (s_1,t_1), (t_2,t_2)\} \).

For any vertex \( v \) in an sp-dag, there is a path from \( s \) to \( t \)
that passes through \( v \). We thus define the \textit{finish vertex} of \( v \) as
the vertex \( u \) which is the closest proper descendant of \( v \) such
that every path from \( v \) to the terminal \( t \) passes through \( u \).

3.1 The sp-dag data structure

Figure 3 shows our data structure for sp-dags. We represent
an sp-dag as a graph \( G \) consisting of a set of vertices \( V \) and
a set of edges \( E \). The data structure assigns to each vertex of
the dag an in-counter data structure, which counts the number
of (unsatisfied) dependencies of the vertex. As the code for
a dag vertex executes, it may dynamically insert and delete
dependencies by using several handles. These handles can
be thought of, in the abstract, as in-counters, but they are
implemented as pointers to specific parts of an in-counter
data structure.
More precisely, a vertex consists of

- a query handle (query), an increment handle (inc), and a decrement handle (dec),
- a flag first_dec indicating whether the first or the second decrement handle should be used,
- a flag dead indicating whether vertex has been executed,
- the finish vertex fin of the vertex, and
- a body, which is a piece of code that will be run when the scheduler executes the vertex.

The sp-dag uses an in-counter data structure Incounter, whose implementation is described in the Section 3.3. The in-counter data structure supplies the following operations:

- make: takes an integer n, creates an in-counter with n as the initial count and returns a handle to the in-counter;
- increment: takes a vertex v, increments its in-counter, and returns two decrement and two increment handles;
- decrement: takes a vertex v and decrements its in-counter;
- is_zero: takes a vertex v and returns true iff that in-counter of v is zero.

Inspired by recent work on formulating a calculus for expressing a broad range of parallel computations [3], our dag data structure provides several operations to construct and schedule daggs dynamically at run time. The operation new_vertex creates a vertex. It takes as arguments a finish vertex u, an increment handle i, a decrement handle d, and a number n indicating the number of dependencies it starts with. It allocates a fresh vertex, sets its dead and first_dec flags to false, and sets the body for the vertex to be a dummy placeholder. It then creates an in-counter and a handle to it. The operation returns the new vertex along with the handle to its in-counter. The handle becomes the query handle. The operation make creates an sp-dag consisting of a root u and its finish vertex z and returns the root u. As can be seen in the function make, the increment and decrement handles of a vertex v always belong to the in-counter of v’s finish vertex.

A vertex u and its finish vertex z can be thought as a computation in which u executes the code specified by its body and then returns to z. This constraint implies that z serially depends on u because z can only be executed after u. As it executes, a vertex u can use the functions chain and spawn to “nest” another sequential or parallel computation (respectively) within the current computation.

The chain operation, which corresponds to serial composition of sp-dags, nests a sequential computation within the current computation. When a vertex u calls chain, the call creates two vertices v and w such that w serially depends on v. Furthermore, v serially depends on u, and z serially depends on w. After calling chain, u terminates and thus dies. The initial in-counters of v and w are set to 0 and 1 respectively (to indicate that v is ready for execution, but w is waiting on

```plaintext
module Incounter = ... (* As defined in Figure 5 *)
module Dag =
G = (V,E)
type handle = Incounter.handle
struct {
  handle query, inc, dec[2];
  boolean first_dec, dead;
  vertex fin;
  (void → void) body;
} vertex;
(∩e, v, handle)
fun new_vertex (vertex u, handle i, handle d, int n) =
  V ← V∪{v}, v ∈ V
  (v.firstDec, v.dead) ← (false, false)
  v.body ← { return }
  v.query ← Incounter.make (n)
  (v.fin, v.inc, v.dec) ← (u, i, d)
  return (v, v.query)
(∩e, v, vertex)
fun make ()
  V ← {v}
  E ← ∅
  (v.firstDec, v.dead) ← (false, false)
  v.body ← { return }
  v.query ← Incounter.make (1)
  u ← new_vertex (u, v.query, v.query, 0)
  return (u, v)
(∩e, v, vertex)
fun chain (vertex u) =
  (w, h) ← new_vertex (u.fin, u.inc, u.dec, 1)
  (v, h) ← new_vertex (w, h, [h, h], ∅)
  E ← E∪{(u, v)}
  u.dead ← true
  return (v, w)
(∩e, v, vertex)
fun spawn (vertex u) =
  (d, i, j) ← Incounter.increment (u.fin)
  (v, _) ← new_vertex (u.fin, i, d, ∅)
  (w, _) ← new_vertex (u.fin, j, d, ∅)
  E ← E∪{(u, v), (u, w)}
  u.dead ← true
  return (v, w)
void
fun signal (u: vertex) =
  Incounter.decrement (u.fin)
  E ← E∪{(u.u.fin)}
```

**Figure 3:** The pseudo-code for our sp-dag data structure.

one other vertex). The edge (u, v) is inserted to the dag to indicate a satisfied dependency between u and v.

The spawn operation, which corresponds to parallel composition of sp-dags, nests a parallel computation with the
current computation. When a vertex $u$ calls spawn, the call creates two vertices, $v$ and $w$, which depend serially on $u$, but are themselves parallel. The operation increments the in-counter of $u$’s finish node, to indicate a serial dependency between $u$’s finish vertex and the new vertices $v$ and $w$. Even though two new vertices are created, the in-counter is incremented once, because one of the vertices can be thought of as a continuation of $u$. The increment operation returns two new increment handles $i$ and $j$ and a decrement-handle pair $d$. The spawn operation then creates the two vertices $v$ and $w$ using the two increment handles, one for each, and the decrement-handle pair $d$ as a shared argument. Assigning each of $v$ and $w$ their own separate increment handles enables controlling contention at the in-counter of $u$’s finish node. As described in Section 3.3, sharing of the decrement handles by the new vertices ($v$, $w$) is critical to the efficiency. As with the chain operation, spawn must be the last operation performed by $u$. It therefore completes by creating the edges from $u$ to $v$ and $w$, and marking $u$ dead.

The operation signal indicates the completion of its argument vertex $u$ by decrementing the in-counter for its finish vertex $v$, and inserting the edge $(u, v)$.

In terms of efficiency, apart from calls to Incounter, the operations of sp-dags involve several simple, constant time, operations. The asymptotic complexity of the sp-dags is thus determined by the complexity of the in-counter data structure.

### 3.2 An Example

Although simple, the sp-dag data structure can be used to create any nested-parallel computation. As an example, we consider the classic parallel Fibonacci program, which computes $n$th Fibonacci number by recursively and in parallel computing the $(n - 1)$th and $(n - 2)$nd numbers. Figure 4 illustrates the code for this example based on the sp-dag data structure and a Scheduler module that implements a standard parallel scheduler, such as work stealing [2, 8]. The function fib_main initializes the system by calling the function Scheduler.initialize and then creates the root and the final vertex of the dag. It then sets the body of the root to run fib, passing the arguments $n$ and result. The vertices root and final are then given to the scheduler to be executed. The scheduler returns when the computation is finished, which leads to the return of the result.

The function fib implements the parallel Fibonacci function. The function start by creating a chain $(u, v)$, passing this_vertex, which is the variable indicating the currently executing vertex, as the parent to the root of the chain. The body of $u$ is set to spawn the two recursive calls and the body of $v$ is set to compute the final result by summing the return values of the recursive calls. Note that when spawning vertices the body of $u$ uses this_vertex as a parent for the vertices created. Since the body is executed when $u$ is executed, at the point of execution this_vertex is equal to $u$.

```plaintext
fib (n: int, dest: int ref) =
if n <= 1 then
  dest ← n
else
  res1 = alloc ()
  res2 = alloc ()
  (u, v) ← Dag.chain (this_vertex)
  u.body ← lambda (). {
    (w1,w2) = Dag.spawn (this_vertex)
    w1.body ← lambda ().fib (n - 1, res1)
    w2.body ← lambda ().fib (n - 2, res2)
    Scheduler.add (w1,w2)
  }
  v.body ← lambda (). *dest ← *res1 + *res2
  Scheduler.add (u, v)

fib_main (n: int) =
int result;
Scheduler.initialize ()
(root, final) ← Dag.make ()
root.body ← lambda ().fib (d, &result)
Scheduler.add (root, final)
return result
```

**Figure 4:** Fibonacci example. The parallel recursive function at the top and the main function, where execution starts, at the bottom.

### 3.3 Incounters

We present the in-counter: a time and space efficient low-contention data structure for keeping track of pending dependencies in sp-dags. When a new dependency for vertex $v$ is created in the dag, its in-counter is incremented. When a dependency vertex in the dag terminates, $v$’s in-counter is decremented.

Our goal is to ensure that the increment and decrement operations are quick; that is, that they access few memory locations and encounter little contention. These goals could seem contradictory at first—if operations access few memory locations then they would conflict often. Our in-counter data structure circumvents this issue by ensuring that each operation accesses few memory locations that are mostly disjoint from those of others.

The in-counter is fundamentally a dynamic SNZI tree. To ensure disjointness of memory accesses, the data structure assigns different handles to different dag vertices. These handles are pointers to SNZI nodes within the in-counter, dictating where in the tree an operation should begin. Thus, if two threads have different handles, their operations will access different memory locations.

The in-counter data structure ensures the invariant that only the leaves have a surplus of zero. This allows us to exploit an important property of SNZI: operations complete when they visit a node with positive surplus, effectively stopping propagation of the change up the tree. Specifically,
module Incounter =
type handle = snzi_node (* A handle is a snzi node *)

(* Growth probability: architecture specific constant. *)
p ← ... (* Make a new counter with surplus n. *)

fun make (n) = snzi_make (n)

(* Auxiliary function: select decrement handle. *)

fun claim_dec (vertex u) =
  if CAS (u.firstDec, false, true) then u.dec[0]
  else u.dec[1]

(* Increment u’s target dependency counter. *)

fun increment (vertex u) =
  (a, b) ← grow (u.inc, p)
  (i1, i2) ← (a, b)
  if u is a left child then d2 ← a
  else d2 ← b
  snzi_arrive (d2)
  d1 ← claim_dec (u)
  return [(d1, d2), i1, i2]

(* Decrement u’s counter. *)

void
d ← claim_dec (u)
snzi_depart (d)

(* Check that u’s counter is zero. *)

boolean
fun is_zero (u) = return snzi_isZero (u.query)

Figure 5: The pseudo-code for the in-counter data structure.

an increment that starts at a leaf completes quickly when it visits the parent. To maintain this invariant, we have to be careful about which SNZI nodes are decremented.

Figure 5 shows the pseudo-code for the in-counter data structure. The in-counter’s interface is similar to the original SNZI data structure: the operation make creates an instance of the data structure and returns a handle to it. The operation increment is meant to be used when a dag vertex spawns. It takes in a dag vertex, and uses its increment handle to increment the SNZI node’s surplus. Before doing so, it calls the grow operation. Intuitively, this growth request notifies the tree of possible contention in the future and gives it a chance to grow to accommodate a higher load. The increment operation returns handles to other nodes in the SNZI tree, that is, two decrement handles and two increment handles, indicating where the newly spawned children of the dag vertex should perform their increment or decrement. The operation decrement is meant to be used when a dag vertex signals the end of its computation. It takes in dag vertex, and uses its decrement handle to decrement the surplus of a previously incremented node.

Since the increment operation is the only thing that can cause growth in the SNZI tree, the structure of the tree is determined by the operations performed on the in-counter structure. The tree is therefore not necessarily balanced. However, as we establish in Section 4, the operations’ running time does not depend on the depth of the tree. In fact, we show that the operations complete in constant amortized time and also lead to constant amortized contention.

To understand the implementation, it is helpful to consider the way the sp-dag structure uses the in-counter operations. The increment operation is called by a dag vertex when the vertex calls the spawn operation, which uses the vertex’s increment handle to determine the node in the SNZI tree, where an arrive operation will start. To find the place, the increment operation first uses a grow operation to check whether the SNZI node pointed at by the handle has children. Additionally, the operation creates new children if necessary (line 20). Intuitively, we create new SNZI nodes to reduce contention by using more space. Thus, if the increment handle has children, we should make use of them and start our arrive operation there. This is exactly what the increment operation does (line 24). Note that, in case the grow operation does not return new children, we simply have the arrive operation start at the increment handle. The increment operation returns two increment handles and two decrement handles; one of each for each of the dag vertex’s new children.

The in-counter algorithm makes use of an important SNZI property: decrements at the top of the tree do not affect any node below them. Furthermore, if there is a depart operation at SNZI node a, it cannot cause a to phase change as long there is surplus anywhere in a’s subtree. These observations lead to the following intuition: to minimize phase changes in the SNZI tree, priority should be given to decrementing nodes closer to the root.

Thus, each dag vertex stores two decrement handles rather than just one. These two decrement handles are shared with the vertex’s sibling, and they are ordered: the first handle always points to a SNZI node that is higher in the tree than the second. Decrement handle is needed by the decrement operation (line 31) and the increment operation. Recall that an increment always returns two decrement handles. One of these decrement handles always points to the SNZI node on which the increment operation started its arrive. However, the other one is inherited from the parent dag vertex (the one that invoked the increment). To preserve the invariant, the incrementing vertex needs to claim a decrement handle (line 25). The two decrement handles returned are always ordered as follows: first, the one inherited from the parent, and then, the one pointing at the freshly incremented SNZI node. In this way, we guarantee that the first decrement handle in any pair points higher in the tree than its counterpart. When a
We first prove that our in-counter data structure is linearizable
and then establish its time and space efficiency. Due to lack of space, we only present the proof for the case where the probability used for the grow operation is 1, i.e., the SNZI tree grows on every increment operation. The proof for different growth probabilities is similar (and leads to contention that depends on the probability given) but more intricate, and is therefore left for the journal version of this paper.

We say that the in-counter is correct if any is_zero operation at the root of the tree correctly indicates whether there is a surplus of increment operations over decrement operations. To prove correctness, we establish a symmetry with the original SNZI data structure. Note that each increment, decrement, and is_zero operation calls the corresponding SNZI operation (i.e., arrive, depart, query respectively) exactly once. We define the linearization points of each operation as that of the corresponding SNZI operation.

Since the correctness condition for the in-counter and for SNZI are the same, we have the following observation.

**Observation 1.** An execution of the in-counter is linearizable if the corresponding SNZI execution is linearizable.

At this point, we recall that any SNZI run is linearizable if there are never more departs than arrives on any node, i.e., surplus at any node is never negative [14]. We show that this invariant indeed holds for any valid execution of the in-counter data structure, and therefore that any valid execution is linearizable. We define a valid execution as follows.

**Definition 1.** An in-counter execution is valid if any handle passed as argument to the decrement operation was returned by a prior increment operation. Furthermore, that handle is passed to at most one decrement operation.

**Lemma 4.1.** Any valid in-counter execution is linearizable.

**Proof.** Observe that every increment operation generates one new decrement handle, and that handle points to the node at which this increment’s arrive operation was called. Thus, in valid executions, any depart on the underlying SNZI data structure (which is always called by the decrement operation) has a corresponding arrive at an earlier point in time. Therefore, there are never more departs than arrives at any SNZI node, and the corresponding SNZI execution is linearizable. From Observation 1, the in-counter execution is linearizable as well. \(\square\)

It is easy to see that any execution of the in-counter that only accesses the object as prescribed by the sp-dag algorithm is valid. From this observation, the main theorem follows immediately.

**Theorem 4.2.** Any execution of the in-counter through sp-dag accesses is linearizable.

4.1 Running Time

To prove that the in-counter is efficient, we analyze shared-memory steps and contention separately: we first show that every operation takes an amortized constant number of shared memory steps, and then we show that each shared memory location experiences constant amortized contention at any time. In our analysis, we do not consider is_zero, since it does not do any non-trivial shared memory steps.

For the analysis, consider an execution on an sp-dag, starting with a dag, consisting of a single root vertex and its corresponding finish vertex. Further, observe that this finish vertex has a single SNZI node as the root of its in-counter. Note that all shared memory steps in the execution are on the in-counter. We begin by making an important observation.

**Lemma 4.3.** There exist at most one increment and one decrement handle pointing to any given SNZI node.

To prove this lemma, note that every increment operation creates new children for the SNZI node whose handle is used, and thus does not produce another handle to that node. A simple induction, starting with the fact that the root node only has one handle pointing to it, yields the result.

We now want to show that every operation on the in-counter performs an amortized constant number of shared memory steps. First, note that the only calls in the in-counter operations that might result in more than a constant number of steps are their corresponding SNZI operations. The SNZI operations do a constant number of shared memory steps per node they reach, but they can be recursively called up an arbitrarily long path in the tree, as long as nodes in that path phase change due to the operation. That is, frequent phase changing in the SNZI nodes will cause operations on the data structure to be slower. In fact, we show that on average, the length of the path traversed by an arrive or depart operation is constant, when those SNZI operations are called only through the in-counter operations.

We say that a dag vertex is live if it is not marked dead and a handle is live if it is owned by a live vertex. We have the following lemma about live vertices.

**Lemma 4.4.** If dag vertex \(v\) is live, then it has not used claim_dec to claim a decrement handle.
Keeping track of live vertices in the dag is important in the analysis, since by Lemma 4.4, these are the vertices that can use their handles and potentially cause more contention.

To show that operations on SNZI nodes are fast, we begin by considering executions in which only increment operations are allowed, and no decrements happen on the in-counter. Our goal is to show that the in-counter’s SNZI tree is relatively saturated, so that arrive operations that are called within a increment only access a constant number of nodes. After we establish that increment operations are fast when decrements are not allowed, we bring back the decrement operations and see that they cannot slow down the increments.

**Lemma 4.5.** Without any decrement operations, when there are no increments mid-execution, only leaves of the in-counter’s SNZI tree can have surplus 0.

**Proof.** The proof is by induction on the number of increment operations called on the in-counter. The tree is initialized with one node with surplus 1. Thus, when there have been 0 increments, no node has surplus 0. Assume that the lemma holds for in-counters that have had up to \( k \) increments on them. Consider an in-counter that has had \( k \) increments, and consider a new increment operation invoked on node \( a \) in the tree. If \( a \) is not a leaf, the increment operation does not create new children for \( a \), and the tree does not grow. Thus, the lemma still holds. If \( a \) is a leaf, the increment operation creates two children for \( a \) and starts an arrive operation on one of the children. As a consequence of the SNZI invariant, we know that the surplus of \( a \) increased by 1. After the increment, regardless of what its surplus was before this operation, \( a \) cannot have surplus 0, and the lemma still holds. □

By Lemma 4.5, we know that, ignoring any effect that decrements may have, the tree has surplus in all non-leaf nodes. Recall that an arrive operation that reaches a node with positive surplus will terminate at that node. Thus, any arrive operation invoked on this tree will only climb up a path until it reaches the first node that was not a leaf before this increment started. That is, since each increment only expands the tree by at most one level, any arrive operation will reach at most 3 nodes.

We now bring back the decrement operations. The danger with decrements is that they could potentially cause non-leaf nodes to phase change back to 0 surplus, meaning that an arrive operation in that node’s subtree might have to traverse a long path up the SNZI tree. Our main lemma shows that traversal of a long path can never happen; if a decrement causes a vertex’s surplus to go back to 0, then no subsequent increment operation will start in that node’s subtree. We present the proof in the full version.

**Lemma 4.6.** If an in-counter node \( a \) at any point had a surplus and then phase changed to 0 surplus, then no live vertices in the dag point anywhere in its subtree.

Note that Lemma 4.5 and Lemma 4.6 immediately give us the following important property.

**Corollary 4.7.** No increment operation can invoke more than 3 arrive operations on the SNZI tree.

**Proof.** We already saw that by Lemma 4.5, if no decrements happen, then each increment operation can invoke at most 3 arrive operations.

Now consider decrements as well. Note that if a non-leaf node has surplus 0 and there is no increment mid-operation at that node, then it must have previously had surplus (again by Lemma 4.5). By Lemma 4.6, no new increment operations can happen on that node’s subtree. Thus, even allowing for decrement operations, no increment operation can invoke more than 3 arrives on the SNZI tree. □

To finish the proof, we amortize the number of depart operations invoked by decrements against the arrives invoked by increments. Recall that the number of depart operations invoked on a SNZI tree is at most the number of arrives invoked on it. We get the following theorem:

**Theorem 4.8.** Any operation performed on the in-counter itself calls an amortized \( O(1) \) operations on the SNZI tree.

We now move on to analyzing the amount of contention a process experiences when executing an operation on the in-counter. Note that there have been several models suggested to account for contention, as discussed in Section 6. In our contention analysis, we say that an operation contends with a shared memory step if that step is non-trivial, and it could, in any execution, be concurrent with the operation at the same memory location. Our results are valid in any contention model in which trivial operations do not affect contention (i.e. such as stalls [16], the CRQW model [21], etc.).

**Theorem 4.9.** Any operation executed on the in-counter experiences \( O(1) \) amortized contention.

**Proof.** We show something stronger: the maximum number of operations that access a single SNZI node over the course of an entire dag computation (other than is_zero operations on the root) is constant.

Consider a node \( a \) in the SNZI tree. By Lemma 4.3, only one increment operation ever starts its arrive at \( a \). That increment has exactly one corresponding decrement whose depart starts at \( a \). By the SNZI algorithm, an arrive at \( a \)’s child only propagates up to \( a \) if that child’s surplus was 0 before the arrive. By Lemma 4.6, this situation can only happen once; if \( a \)’s child’s surplus returns to 0 after being higher, the counter will never be incremented again. Thus, each of \( a \)’s children can only ever be responsible for at most two operations that access \( a \); the initial arrive and the final depart from that subtree.

In total, we get a maximum of 6 operations that access \( a \) over the course of the computation. Note also that the first
We report an empirical comparison between our in-counter algorithm and an atomic, fetch-and-add counter because the fetch-and-add counter is optimal for such operations. Therefore, any operation on the in-counter can be concurrent with at most 4 other operations per SNZI node it accesses. By Theorem 4.8, every operation on the in-counter accesses amortized $O(1)$ nodes. Therefore, any such operation experiences amortized $O(1)$ contention.

4.2 Space bounds

Shrinking the tree by removing unnecessary nodes is tricky because, in its most general form, an ideal solution likely requires knowing the future of the computation. Knowing where in the tree to shrink requires knowing which nodes are not likely to be incremented in the future. For the specific case of the $\text{grow}$ probability of 1, we establish a safety property that makes it possible to keep the data structure compact by removing SNZI nodes that correspond to deleted sub-graphs of the sp-dag. Due to space constraints, we present the details in the full version [1].

5. Experimental evaluation

We report an empirical comparison between our in-counter algorithm, the simple fetch-and-add counter, and an algorithm using fixed-size SNZI trees. Our implementation consists of a small, C++ library that uses a state-of-the-art implementation of a work-stealing scheduler [2]. Overall, our results show that the simple, fetch-and-add counter performs well only when there is one core, the fixed-size SNZI trees scale better, but our in-counter algorithm outperforms the others when the core count is two or more.

Implementation. Our implementation of core SNZI operations $\text{snzi\_arrive}$ and $\text{snzi\_depart}$ follows closely the algorithm presented in the original SNZI paper, except for two differences. First, we do not need $\text{snzi\_query}$ because readiness detection is performed via the return result of $\text{snzi\_depart}$. This method suffices because only the caller of $\text{snzi\_depart}$ can bring the count to zero. Second, our $\text{snzi\_depart}$ returns true if the call brought the counter to zero. To control the grain of contention, we use the probabilistic technique presented in Section 2, with probability $p := \frac{1}{25c}$, where $c$ is the number of cores. The idea of using $c$ is to try to keep contention the same as the number of cores increase. We chose the constant factor 25, because it yields good results, but as our Threshold study, described below, shows many other constants also yield good results, e.g., any constant in the range $2.5c \leq p \leq 25c$ yields qualitatively the same results. The implementation of the sp-dags and in-counter data structures (Section 3) build on the SNZI data structure. The sp-dags interface enables writing nested-parallel programs, using various constructs, such as fork-join and async-finish; we use the latter in our experiments.

We compare our in-counter with an atomic, fetch-and-add counter because the fetch-and-add counter is optimal for very small numbers of cores. For higher number of cores, we designed a different, SNZI-based algorithm that uses a fixed-depth SNZI tree. This algorithm gives us another point of comparison, by offering a data structure that uses the existing state of the art more directly. The fixed-depth SNZI algorithm allocates for each finish block a SNZI tree of $2^d + 1$ nodes, for a given depth $d$. To maintain the critical SNZI invariant that the surplus of a SNZI node never becomes negative, the fixed-depth SNZI algorithm ensures that every $\text{snzi\_depart}$ call targets the same SNZI node that was targeted by a matching $\text{snzi\_arrive}$ call. To determine which SNZI node to be targeted by a $\text{snzi\_arrive}$ call, we map DAG vertices to SNZI nodes using a hash function to ensure that operations are spread evenly across the SNZI tree.

Experimental setup. We compiled the code using GCC -O2 -march=native (version 5.2). Our machine has four Intel E7-4870 chips and 1Tb of RAM and runs Ubuntu Linux kernel v3.13.0-66-generic. Each chip has ten cores (40 total) and shares a 30Mb L3 cache. Each core runs at 2.4Ghz and has 256Kb of L2 cache and 32Kb of L1 cache. The machine has a non-uniform memory architecture (NUMA), whereby RAM is partitioned into four banks. Pages are, by default in our Linux distribution, allocated to banks according to the first-touch policy, which assigns a freshly allocated page to the bank of the processor that first accesses the page. An alternative policy assigns pages to banks in a round-robin fashion. We determined that, for the purposes of our experiments, the choice of NUMA policy has no significant impact on our main results. The experiment we performed to back this claim is described in more detail in the full paper [1]. For all other experiments, we used the round-robin policy.

Benchmarks: fanin and indegree2. To study scalability, we use the two small benchmarks shown in Figures 6 and 7. The fanin benchmark performs $n$ async calls, all of which synchronize at a single finish block, making the finish block a potential source of contention. The fanin benchmark implements a common pattern, such as a parallel-for, where a number of independent computations are forked to execute in parallel and synchronize at termination. Our second benchmark, indegree2 implements the same pattern as in fanin but by using binary fork join. This program yields a computation dag in which each finish vertex has indegree 2.
Scalability study. Figure 8 shows the scalability of the fanin benchmark using different counter data structures. With one core, the Fetch & Add counter performs best because there is no contention, but with more, Fetch & Add gives worst performance for all core counts (this pattern is occluded by the other curves). For more than one core, our in-counter performs the best. The fixed-depth SNZI algorithm scales poorly when depth is small, doing best at the tree depth of 8, where there are enough nodes to deal with contention among 40 cores. Increasing the depth further does not improve the performance, however.

Size-invariance study. Theorem 4.9 shows that the amortized contention of our in-counter is constant. We test this result empirically by measuring the running time of the fanin benchmark for different input parameters $n$. As shown in Figure 9, for all input sizes considered and for all core counts, the throughput is within a factor 2 of the single-core (no-contention) Fetch & Add counter. The throughput suffers a bit for the smaller values of $n$ because, at these values, there is not enough useful parallelism to feed the available cores.

Low-indegree study. Our next experiment examines the overhead imposed by the in-counter by using the indegree2 benchmark shown in Figure 7. The results, illustrated in Figure 10, show that our in-counter data structure is within a factor 2 of the best performer, the fetch-and-add counter. For SNZI, we only considered small-depths, since larger ones took too long to run. With the fixed-depth SNZI, the benchmark creates many finish blocks and thus many SNZI trees, becoming inefficient for large trees.

Threshold study. In Section 2, we presented a technique to control the grain of contention by expanding the SNZI tree dynamically. Here, we report, using the fanin benchmark, the results of varying the range of probabilities $p = \frac{1}{\text{threshold}}$ for different settings of threshold. As shown in Figure 11, essentially any threshold between 50 and 1000 works well. We separately checked that on our test machine using a constant threshold such as 100 or 1000 yields good results when using any number of cores. On a machine, perhaps with many more cores, the best setting might be different or might also depend on the number of cores used.

6. Related Work

Upper bounds for several non-blocking data structures, accounting also for contention, have been proven for several tree- and list-based search structures [13, 18, 39]. In these upper bounds, contention factors in as a linear additive term, which is consistent with established lower bounds for many problems in the general concurrency model [15, 16, 27, 30]. In contrast, for relaxed counters, we show that contention for series-parallel programs can be upper bounded by a constant.

Complexity models for analyzing contention and techniques for designing algorithms for reducing contention has also been important subject of study [12, 20, 21, 31, 36]. Contention has also been identified as an important practical
performance issue in multiprocessor systems. Techniques that reduce contention by detecting and introducing wait periods or using tokens to be passed between threads proved to be essential for reducing detrimental effects of contention \cite{4,6}. Managing contention in software-transactional memory may require contention managers that schedule transactions carefully to minimize aborted transactions \cite{23,26,40,41}.

In this paper, we consider non-blocking, or lock-free algorithms. Since our upper bounds are quite good, $O(1)$, our algorithms guarantee that each operation completes quickly in our relaxed concurrency model. Wait-free data structures can guarantee similar properties in the general concurrency model \cite{24,25}. Such data structures were traditionally considered to be impractical, though recent results show that they may be also be practical \cite{33,44}.

7. Conclusion

The design and analysis of provably efficient concurrent and specifically non-blocking data structures has been a challenging problem. One reason is the complexity of the concurrency models. Another reason is the difficulty of accounting for important practical concerns, such as contention, that can significantly impact performance. In this paper, we show that it is possible to design provably efficient data structures under a slightly more restrictive but still important concurrency model: nested-parallel computations. Our results take advantage of certain structural invariants in nested parallel computations, specifically those that concern the creation, termination, and synchronization of threads. Interesting future research directions include the design and analysis of other concurrent data structures for the nested-parallel concurrency model and consideration of more general, but still restricted, models of concurrency, such as those based on futures.

Acknowledgments

This research is partially supported by the a fellowship from Natural Sciences and Engineering Research Council of Canada (NSERC), and grants from National Science Foundation (CCF-1320563 and CCF-1408940) and European Research Council (grant ERC-2012-StG-308246). We are grateful to Doug Lea for his feedback on the paper.

References


A. Running Time

Lemma A.1. If an in-counter node \( a \) at any point had a surplus and then phase changed to 0 surplus, then no live vertices in the dag point to \( a \) or its subtree.

Proof. We prove this by induction on the size of \( a \)'s subtree. Assume that node \( a \) in the in-counter has a surplus at time \( t \), and consider the decrement operation that caused \( a \)'s surplus to become 0.

If \( a \) has no children, we only need to consider the handles that point directly at \( a \), since it has no subtree. Note that it must be the case that \( a \) was incremented by an arrive operation that started there, in a increment operation that had an increment handle to \( a \)'s parent. During that operation, a decrement handle to \( a \) was created, and placed as the second of the pair of decrement handles returned. If \( a \) was now decremented, both dag vertices that shared these decrement handles must have claimed them, and thus, by Lemma 4.4, neither of them is now live. In particular, this means that the dag vertex that owned the increment handle to \( a \) is dead. So, no live dag vertex has a handle to \( a \).

Assume that the lemma holds for nodes with subtrees of size at most \( k \). Now consider a SNZI node \( a \) which has a subtree of size \( k + 1 \). Consider the decrement operation that caused \( a \)'s surplus to change to 0. There are two possible cases: either (1) this decrement’s first depart is at \( a \), or (2) it started at one of \( a \)'s descendants.

Case 1. This decrement started at \( a \). Note that \( a \) has children, but neither of them has a surplus at the time this decrement operation starts, since otherwise, \( a \) would have a surplus due to its children, and would not phase change. Note that for \( a \) to have children, an increment operation must have started at \( a \) and used touch to create \( a \)'s children. This same increment then arrived at one of \( a \)'s children. So at least one of \( a \)'s children must have had surplus at some point, and now neither of them do. By the induction hypothesis, if both of them had surplus at some point, then no live dag handles point anywhere in \( a \)'s childrens’ subtrees now. Furthermore, since by Lemma 4.3 only one increment handle and one decrement handle are ever created for \( a \), no handle in the dag points anywhere in \( a \)'s subtree (including \( a \) itself).

If only one of \( a \)'s children ever had surplus, by the induction hypothesis, that child now has no handles pointing to \( a \) or its subtree. Furthermore, for that child’s surplus to become 0, both of the dag vertices that were created in the spawn operation that created \( a \)'s children must have used their decrement handles. Note that this includes the only dag vertex that had an increment handle to \( a \)'s other child. Thus, neither of \( a \)'s children have any live handles pointing to them, and since \( a \)'s handles have been used, no live handles now point to anywhere in \( a \)'s subtree.

Case 2. The decrement that caused \( a \)'s phase change started in \( a \)'s subtree, but not at \( a \) itself. Let \( a_l \) denote \( a \)'s right child and \( a_l \) denote its left child. Without loss of generality, assume that the decrement started in \( a_l \)'s subtree.

First note that \( a_l \) must have had surplus before this decrement started, since no node can have a negative surplus. We also know that after this decrement, \( a_l \) has a surplus of 0, since it must have phase changed in order for a depart to reach \( a \). Thus, by the induction hypothesis, no live handles point to \( a_l \).

We now only have to consider \( a_l \) to see whether there are still live handles pointing to that subtree. We know that \( a_l \) does not have surplus, since otherwise \( a_l \) would not phase change due to this decrement. Assume by contradiction that there is a live handle pointing to \( a_l \). Then by the induction hypothesis, \( a_l \) has never had a surplus, and there is a live increment handle to it. Note that, by the algorithm, the dag vertex that has the increment handle to \( a_l \) must have a decrement handle pointing to \( a_l \). By Lemma 4.4, this decrement pointer has never been used, thus contradicting our earlier conclusion that \( a_l \) has 0 surplus. \( \square \)

B. Space bounds

We show that our in-counter data structure is compact by establishing a safety property that makes it possible to remove SNZI nodes that correspond to deleted sub-graphs of the sp-dag. Due to "space" constraints, we present the details in the appendix.

We would like to show that our in-counter data structure does not take much more space than the dag computation uses on its own. To do so, we will show when it is safe to delete SNZI nodes in relation to the state of the dag vertices that have handles to them. First, however, we note that even without the dynamic shrinking of the SNZI tree, there are never more nodes in the in-counter than the total number of dag vertices created in an execution. This is because whenever a dag vertex spawns to create two more children, the SNZI tree also grows by two nodes.

However, we can do better than that by deleting SNZI nodes that will never be used again. For this, we begin with the following lemma:

Lemma B.1. Any node whose surplus was positive and then returned to 0 may be deleted safely from the SNZI tree.

This lemma is a direct consequence of Lemma 4.6. Any node whose surplus goes back to 0 has no live handles pointing to it, and is thus safe for deletion.

Using this knowledge, we can shrink the SNZI tree as the computation progresses. To be able to express the next theorem, we need to define when a vertex \( u \) in the dag has finished. This is a recursive definition:

- Base Case. If \( u \) has no children, then it is finished when it signals the end of its computation.
• **Iterative Case.** If \( a \) has children, then is is finished when both of its children have finished.

We note the following important lemma which relates finished vertices to the state of the execution in an sp-dag that uses in-counters.

**Lemma B.2.** If a vertex \( u \) has finished, then the decrement handle that it claimed has been used.

**Proof.** We prove this by induction on the number of descendants \( u \) has. If \( u \) doesn’t have any descendants, then we know it had to use its claimed decrement handle before it finished, since by definition, it finishes when it signals, and to signal, it needs to call the decrement operation on the in-counter.

Assume the lemma holds for any dag vertex \( u \) with up to \( k \) descendants. Let \( u \) be a dag vertex with \( k + 1 \) descendants. By the induction hypothesis, both of its children have used their claimed decrement handles. Note that of those two handles, one was claimed by \( u \) and passed down to its children. Thus, \( u \)'s decrement handle has been used.

Now, we can relate finished vertices to deallocation.

**Theorem B.3.** Consider a dag vertex \( u \) and its increment handle, pointing to SNZI node \( a \). If \( u \) has finished, then any node in \( a \)'s subtree, excluding \( a \) itself, can be safely deleted from the SNZI tree.

**Proof.** We prove this by induction on the size of \( a \)'s subtree. If \( a \) doesn’t have children, then we are done - no nodes are deleted when \( u \) finishes.

Assume that if \( u \) is finished, and has an increment handle to SNZI node \( a \), which has a subtree of size at most \( k \), then \( a \)'s entire subtree, excluding \( a \) itself, can be deleted. Let \( u \) be a finished vertex whose increment handle points to a node \( a \) with a subtree of size \( k + 1 \).

Note that by definition, if \( u \) is finished, then both of its children are finished as well. Note that, by the algorithm, \( u \)'s children’s increment handles must point to \( a \)'s children in the SNZI tree. Thus, by the induction hypothesis, \( a \)'s entire subtree, excluding itself and its children, can be deleted safely. In particular, it means that neither of \( a \)'s children have surplus due to their children. Note that only one of \( a \)'s two children had an arrive operation start on it (this was done by \( u \)'s increment operation, which created \( a \)'s children). Thus, if neither of them has surplus due to their children, at most one of them has surplus. Note that the decrement handle to that child was owned by one of \( u \)'s children. By Lemma B.2, since both of \( u \)'s children have finished, they have both used their decrement handles. Thus, \( a \)'s child has been decremented, and now has surplus 0. Furthermore, even if \( a \)'s other child never had surplus, the dag vertex that owned its increment handle has used its decrement handle, so \( a \)'s other child can never grow. Thus, both of \( a \)'s children can be safely deleted. \( \square \)

**Figure 12:** Reproduction of the original SNZI experiment that is presented by Figure 10 of the original SNZI paper [14].

**C. Supplementary experiments**

**C.1 SNZI reproduction study**

For our experiments, we implemented the SNZI algorithm for the increment and decrement operations as described in the original SNZI paper [14]. To gain confidence in this implementation and check that it performs as expected, we first performed a reproduction study. The reproduction study reproduces the results shown in Figure 10 of the original SNZI paper [14], but uses our own implementation and our 40-core test machine, which is slightly smaller than the machine used in the original paper, and also has a quite different architecture. Ours is a 4 node, 10 core Intel x86 machine, whereas the machine from the original paper is a 24-node, dual core Sun Sparc.

The plot in Figure 12 shows the results. We see that the best setting for the SNZI tree height is 4, whereas in the original SNZI study, on the authors’ 48-core test machine, the best setting is 5. This difference is consistent with the different processor counts of the machines. Our fetch-and-add scales significantly better than theirs, which can be attributed to the difference between the more tightly coupled nature of our machine (4 nodes versus 24 nodes). Our best setting achieves almost the same throughout as the original: approximately 1000 operations per millisecond (note that the Figure reports throughput in seconds). Furthermore, the trends for varying numbers of cores/processors are similar. For example, in both studies the worst performer after just four cores is the single-cell, fetch-and-add algorithm and that, on 40/48 cores, the best performing setting of SNZI outperforms the single-cell, fetch-and-add algorithm by at least an order of magnitude.

**C.2 Disabling NUMA**

To determine whether NUMA has a significant effect on the performance of the counter structures, we ran the experiment shown in Figure 8, but this time with NUMA disabled. The result of this experiment shown in Figure 13. What this experiment demonstrates is that the NUMA configuration has no significant effect on performance.
C.3 Granularity study.

The benchmarks we considered so far were written so that, when they run, they perform no useful work: all they do is spawn and synchronize DAG vertices. We chose this because it simulates a highly concurrent environment, where the number of processors can be larger than what is available to us today, making it possible to determine the effects of contention at larger scale. In the study presented in this section, we simulated a wider range of workloads on modern machines by controlling the granularity of our benchmark program. In particular, to set up this experiment, we modified our benchmark program in Figure 6 so that the program performs some dummy work each time it makes a leaf-level call to `fanin`. Figure 14 shows the results of this experiment on the correspondingly modified `fanin` program. Looking at the horizontal axis, note that each unit of dummy work takes approximately one nanosecond on our test machine. Here, we see that the Fetch & Add cell is much slower until the amount of dummy work per task reaches 100 microseconds. We confirmed that at higher amounts of per-task dummy work, all algorithms perform essentially the same.

On a machine such as ours, a 100 microseconds is a relatively large granularity, with the desired granularity usually ranging between 10-50 microseconds. Note that at 10 microseconds, there is a noticeable gap between Fetch & Add counter, and relaxed counters such as ours and SNZI. This result suggests that the choice of counter algorithm used by a fork-join system has a substantial impact, even on a tightly coupled machine with just 40 cores.

It is not too hard to see that, as the number of cores on multicore machines increase, the impact of the counter algorithm is only going to increase. The reason is twofold. First, as the number of cores increases, so does the impact of contention. Second, the usefulness of having even a small (i.e., 10-50 microsecond) task-grain size is not likely to change much as machines with larger core counts become available, because grain size is selected to amortize the cost of task creation, which, for all practical purposes, is tightly aligned with memory access times and is largely independent of the number of processors.

To understand the impact of more contention on performance, we performed several additional experiments. The results are shown in Figures 15a through 15e. Each of these plots shows the speedup of the benchmark with varying the number of cores. For the baseline measurement, we use Fetch & Add on one core. The trends show that, as the number of cores increases, so does the impact of the counter algorithm. Furthermore, as the workload increases, the difference between the algorithms diminishes, but even for such a large amount of per-task dummy work as 10 microseconds, the difference is still noticeable.

D. Artifact description

D.1 Abstract

This artifact contains a script that, when run, repeats the experiments reported in our PPoPP’17 paper titled Provably Efficient Low Contention Non-Blocking Data Structures for Parallel Computation. The script repeats all experiments reported in the paper, that is, those shown in Figures 8, 9, and 10, and most that appear in the appendix, excluding our reproduction of the SNZI experiment (Figure 12) and our NUMA study (Figure 13). The result of the script are plots (and raw execution times). Each plot has a counterpart in the paper and should represent the same experimental setup. To repeat our experiments, obtain the necessary machine and software dependencies, run the script `go.sh`, and check the output. The default setting for the maximum number of processors to be used is 40, but this setting can be customized as described below.

D.2 Description

D.2.1 Check-list (artifact meta information)
- Program: C++ code
Figure 15: Fanin benchmark with $n = 8$ million, & varying number of cores. Higher is better.
• Compilation: gcc-6.1+
• Binary: x86_64 executable
• Run-time environment: Ubuntu 14.0+
• Hardware: Any multicore x86_64 machine. We recommend at least 40 cores.
• Output: pdf plots in the same format as presented in the paper and raw results files showing among other data the wall-clock running times.
• Experiment workflow: Create a new folder; in that folder, run the supplied script go.sh; find results in new folder named Results.
• Experiment customization: To deal with noise, our benchmark scripts are configured so that each run is repeated multiple times. The default number of times is 30, and every value reported in the plots represents the average of the runs. With this configuration, on our test machine, the time to complete the experiments in the paper is about six hours. The time including supplementary benchmarks is about eleven hours. If you prefer to change this setting, in the script go.sh, change the nb_runs variable to the desired number. The maximum number of cores to be used by the experiment can be set by passing to the benchmarking script the argument -max_proc p, where p represents the desired number of processors.
• Publicly available?: Yes, all source code is hosted by github. The github addresses may be found in the go.sh script.

D.2.2 How delivered
The go.sh script can be found at the following url: http://gallium.inria.fr/~rainey/go.sh. Our benchmark software is hosted with code on github.

D.2.3 Hardware dependencies
Our benchmarks are configured to run on x86_64 machines.

D.2.4 Software dependencies
• Ocaml 4.0+
• hwloc (any recent version)
• gperftools (any recent version; used to provide tcmalloc, scalable memory allocator)
• R (any recent version; used to generate plots)

D.3 Installation
First, create a new folder in which to run the experiments. Then, put in that folder an executable copy of run script go.sh.

D.4 Experiment workflow
Run the script go.sh. After it completes, a folder named Results should appear. This folder contains the results of the experiment.

D.5 Evaluation and expected result
The Results folder should contain three pdf and three text files. The plots in the files plots_proc.pdf, plots_size.pdf, and plots_indegree2.pdf correspond to the plots shown in Figures 8, 9, and 10. The results files store the experimental data in an ad hoc text file format. The format is human readable. For example, in the following snippet, the key-value pairs that appear before the “—” denote input and what follows output. In this run, we are, for example, using a single core, threshold set to 40000, and n to 16777216. The running time reported by the benchmark is 4.235 seconds.

==========
machine rainey-Precision-T1700
prog ./counters.virtual
bench fanin
algo dyn
proc 1
threshold 40000
n 16777216
---
timeout 400
exectime 4.235
nb_promotions 0
nb_steals 0
nb_stacklet_allocations 0
nb_incounter_nodes 415
killed 0
==========
...

D.6 Experiment customization
It is possible to run some of the supplementary experiments that we report in the appendix. To run these experiments, run the same command as before, but with an argument: go.sh appendix. The result will be the same plots as before in the same directory, plus some new ones. The first is the threshold experiment in Section 5. The plot named plots_threshold.pdf represents the same experiment as the plot in Figure 11. The second set of plots come from the granularity study in Section C.3. The file named plots_workload_fanin.pdf represents the same experiment as the plot in Figure 14. Finally, the file named plots_workload_fanin_speedup*.pdf represents the same experiments as are shown in the plots in Figures 15a - 15e.