Canonical basis of commuting diagrams

Application to the mechanical synthesis of coherence conditions

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Background

1. Category theory

a. A category with finite products admits the following natural isomorphisms:

$$\text{Mon}\begin{cases} Ass(\times): & (A\times B)\times C = A\times(B\times C) \\ Idl(1,\times): & 1\times A = A \\ Idr(1,\times): & A\times 1 = A \end{cases}$$

Furthermore, these isomorphisms verify a strong compatibility condition:

coherence condition

For every object A, and every morphism $f : A \to A$ formed by composition of the above isomorphisms and their inverses, we have $\vdash f = Id_A$.

b. Abstracting: distinguished object 1, bifunctor $\times$, isomorphisms verifying Mon and the coherence condition:
$\to$ monoidal category.
More generally: "algebroidal" categories.

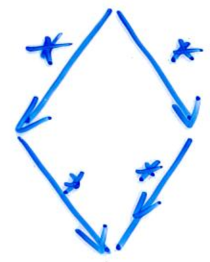
c. Mac Lane and Kelly gave sufficient conditions for the coherence condition, expressed in the form of a finite basis of commuting diagrams.

2. Computer science : The Knuth & Bendix method

(a). Equational theories – Birkhoff's theorem
Automated deduction with equalities.

(b) Simplification – Term rewriting systems.

(c) Newman's lemma . 1942.



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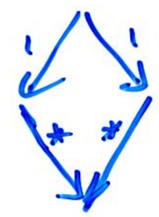


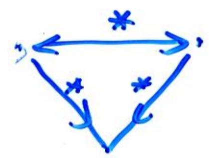
i.f. → w.f.

confluence

local confluence

(Noetherian induction).

(d). Church-Rosser property



≡ confluence

⇒ uniqueness of normal forms

i.f. → w.f. existence of canonical forms.

(e) Unification – Existence of unit basis of solution in completely free algebras (principal unifiers)
Herbrand 1930 Robinson 1964

(f) Knuth & Bendix. 1969.

- Superposition algorithm.
- Decision procedure for local confluence of t.r.s.
- Completion of equational presentation into canonical sets of rewrite rules.

(g) Peterson – Stickel 1975. Extension to C&A.

3. The intuitionism paradigm: keeping the information

Keeping the justification for confluence = keeping the arrows (natural transformation composition)
⇒ extending confluence diagram into commuting diagrams.

N.B. congruence of rewriting = functoriality
substitutivity = naturalness

ⓐ Well-founded presentation of natural transformations as oriented version of algebroidal theories.
$\exists$ w.f. ordering $>$ on objects s.t.
- every instance of nat. trans. $T : A \to B \Rightarrow A > B$
- functors preserve the ordering.

ⓑ Simplifying arrow
- generators : $F(\tau_{A_1 \dots A_n})$
- G^* → simplifying arrow of length n.

ⓒ We call pre-diagram a pair of two simplifying arrows with the same domain: $f : A \to B$ $g : A \to C$, which we write:

$$\langle f, g \rangle : A \longrightarrow B \overset{?}{=} C$$

The pre-diagram is said to be local iff both f and g are of length 1.

We say the (pre-)diagram is commutable iff we can complete it by simplifying arrows into a commuting diagram, i.e. iff there exists an object D and simplifying arrows $f' : B \to D$ and $g' : C \to D$ such that $f ; f' = g ; g'$.

ⓓ
- locally commuting systems of w.f. nat. transf.
- commuting " " "

(e) **Newman's lemma** (the categorical version)

Every locally commuting system is commuting.

(f) The superposition algorithm defines, for every finite system of natural transformations, a finite set of pre-diagram schemas called critical pre-diagrams.

Knuth-Bendix theorem (the categorical version)
A system of w.f. natural transformations is locally commuting iff every critical pre-diagram is commutable.

Example The Mon presentation.
We get 5 critical pre-diagrams:

$$\langle Ass_{A\times B,C,D},\ Ass_{A,B,C}\times D\rangle : ((A\times B)\times C)\times D \to (A\times B)\times(C\times D) \overset{?}{=} (A\times(B\times C))\times D$$

$$\langle Ass_{1,B,C},\ Idl_B\times C\rangle : (1\times B)\times C \to 1\times(B\times C) \overset{?}{=} B\times C$$

$$\langle Ass_{A,1,C},\ Idr_A\times C\rangle : (A\times 1)\times C \to A\times(1\times C) \overset{?}{=} A\times C$$

$$\langle Ass_{A,B,1},\ Idr_{A\times B}\rangle : (A\times B)\times 1 \to A\times(B\times 1) \overset{?}{=} A\times B$$

$$\langle Idl_1,\ Idr_1\rangle : 1\times 1 \to 1 \overset{?}{=} 1$$

We generate coherence conditions as justifications of the commutability of each critical pre-diagram, by keeping a trace of the Knuth-Bendix computation.

end result vs. trace.

Example

$$\begin{tikzcd}
 & ((A\times B)\times C)\times D \arrow[dl, "Ass"'] \arrow[dr, "Ass"] & \\
(A\times B)\times(C\times D) \arrow[ddr, "Ass"'] & \text{Diag 1} & (A\times(B\times C))\times D \arrow[d, "Ass"] \\
 & & A\times((B\times C)\times D) \arrow[dl, "Ass"'] \\
 & A\times(B\times(C\times D)) &
\end{tikzcd}$$

Diag 1: $Ass_{A\times B,C,D} ; Ass_{A,B,C\times D} = Ass_{A,B,C}\times D ; Ass_{A,B\times C,D} ; A\times Ass_{B,C,D}$

Diag 2: $Ass_{1,B,C} ; Idl_{B\times C} = Idl_B \times C$

Diag 3: $Ass_{A,1,C} ; A\times Idl_C = Idr_A \times C$

Diag 4: $Ass_{A,B,1} ; A\times Idr_B = Idr_{A\times B}$

Diag 5: $Idl_1 = Idr_1$

Application: the Mac-Lane – Kelly coherence condition.

The Church-Rosser lemma. Assume we have a commuting system of w.f. nat. transf., and let $f : A = B$ be an arbitrary isomorphism of the theory (using perhaps the inverses). Then there exists object C and simplifying arrows $g : A \to C$ and $h : B \to C$ such that $g = f ; h$.

The coherence theorem. Assume a theory whose presentation is such that all the critical pre-diagrams are commutable. Then the theory is coherent.

Z commuting is stronger than coherent

Mac Lane gives only Diag 1, Diag 3 and Diag 5.

What's new ?

- The method is completely mechanizable, by tracing the Knuth-Bendix computation.
- It extends to the idea of completing a non commuting theory into a commuting one.
- It can be extended to Commutative - Associative theories.

Symmetric isomorphism:

$Comm(X) : A \times B \to B \times A$.

We factor out Comm and Ass as permutation isomorphisms, and compute commuting diagrams modulo those permutations.

Example: Symmetric monoidal categories

- We drop Idr.
- We check that the theory is commuting modulo permutations, which gives a finite basis of diagrams.

Z. Now coherence modulo permutation must be expressed as follows:

Every $f : A \to A$ obtained by composition of the isomorphisms and their inverses is a permutation (and not in general an identity).

e.g. $\langle Snd, Fst \rangle : A \times A \cong A \times A$

This strictly generalizes the known result concerning diagram schemas whose object expressions are linear.

?? Should we compile an encyclopedia of basis of commuting diagrams