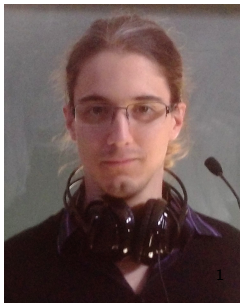


An introduction to polarised L calculi

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Séminaire Gallium, October 15th 2018



Why?

L: polarized versions of the $\lambda\mu\tilde{\mu}$ abstract-machine calculi.

Claim: a good **syntax** to talk about

- classical computation
- mixed-polarity equivalences
- effects
- purity

(Semantics: roughly in line with CBPV models.)

Related works:

- Zeilberger (direct relation to focusing; no syntax)
- CBPV (related models, worse syntax)

Section 1

$\mu\tilde{\mu}$ recap

Abstract machines

$$\begin{array}{l} \langle t \ u \ \| \ e \rangle \triangleright_{\text{abs}} \langle t \ \| \ u \cdot e \rangle \\ \langle \lambda x. t \ \| \ u \cdot e \rangle \triangleright_{\text{abs}} \langle t \ [u/x] \ \| \ e \rangle \end{array}$$

Abstract machines

$$\begin{array}{l} \langle t \ u \ \| \ e \rangle \triangleright_{\text{abs}} \langle t \ \| \ u \cdot e \rangle \\ \langle \lambda x. t \ \| \ u \cdot e \rangle \triangleright_{\text{abs}} \langle t \ [u/x] \ \| \ e \rangle \end{array}$$

$c ::= \langle t \ \| \ e \rangle$ configuration

$t, u ::=$

x, y, z	variable
$t \ u$	application
$\lambda x. t$	λ -abstraction

$e, f ::=$

\star	empty
$t \cdot e$	application stack

Introducing μ

$$\langle t \ u \parallel e \rangle \triangleright_{\text{abs}} \langle t \parallel u \cdot e \rangle$$

This reduction **defines** $(t \ u)$:

$$\langle t \ u \mid _ : _ \mid e \rangle \longmapsto \langle t \parallel u \cdot e \rangle$$

It is the term that, when put against $\mid e \rangle$, reduces to $\langle t \parallel u \cdot e \rangle$.

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Idea : introduces a **universal** solution to this problem

$$\mu\alpha. \langle t \parallel u \cdot \alpha \rangle$$

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Idea : introduces a **universal** solution to this problem

$$\mu\alpha. \langle t \parallel u \cdot \alpha \rangle$$

$$\langle \mu\alpha. c \parallel e \rangle \triangleright_{\mu} c [e/\alpha]$$

Introducing μ

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Idea : introduces a **universal** solution to this problem

$$\mu\alpha. \langle t \parallel u \cdot \alpha \rangle$$

$$\langle \mu\alpha. c \parallel e \rangle \triangleright_{\mu} c [e/\alpha]$$

$$t \ u \stackrel{\text{def}}{:=} \mu\alpha. \langle t \parallel u \cdot \alpha \rangle$$

Detroducing λ

$$\langle \lambda x. t \parallel u \cdot e \rangle \triangleright_{\text{abs}} \langle t [u/x] \parallel e \rangle$$

Idea : λ pattern-matches on the context, deconstructing $u \cdot e$.

Detroducing λ

$$\langle \lambda x. t \parallel u \cdot e \rangle \triangleright_{\text{abs}} \langle t [u/x] \parallel e \rangle$$

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$$\mu(x \cdot \alpha). c$$

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$$\mu(x \cdot \alpha). c$$

$$\langle \mu(x \cdot \alpha). c \parallel u \cdot e \rangle \triangleright_{\text{fun}} c [u/x, e/\alpha]$$

Detroducing λ

$$\langle \lambda x. t \parallel u \cdot e \rangle \triangleright_{\text{abs}} \langle t [u/x] \parallel e \rangle$$

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$$\mu(x \cdot \alpha). c$$

$$\langle \mu(x \cdot \alpha). c \parallel u \cdot e \rangle \triangleright_{\text{fun}} c [u/x, e/\alpha]$$

$$\lambda x. t \stackrel{\text{def}}{=} \mu(x \cdot \alpha). \langle t \parallel \alpha \rangle$$

Other datatypes and $\tilde{\mu}$

$c ::= \langle t \parallel e \rangle$ command

$t, u ::=$

| x, y, z
| $\mu\alpha. c$
| $\mu(x \cdot \alpha). c$
| (t, u)

$e, f ::=$

| \star, α, β
|
| $t \cdot e$

$\langle \text{let } (x_1, x_2) = t \text{ in } u \parallel e \rangle \triangleright_{\text{abs}} \langle t \parallel \text{let } (x_1, x_2) = \square \text{ in } \langle u \parallel e \rangle \rangle$

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$\text{let } (x_1, x_2) = t \text{ in } u \stackrel{\text{def}}{:=} \mu\alpha. \langle t \parallel \tilde{\mu}(x_1, x_2). \langle u \parallel \alpha \rangle \rangle$

Other datatypes and $\tilde{\mu}$

$c ::= \langle t \parallel e \rangle$ command

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| x, y, z
| $\mu\alpha. c$
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| \star, α, β
|
| $t \cdot e$
| $\tilde{\mu}(x_1, x_2). c$

$\langle \text{let } (x_1, x_2) = t \text{ in } u \parallel e \rangle \triangleright_{\text{abs}} \langle t \parallel \text{let } (x_1, x_2) = \square \text{ in } \langle u \parallel e \rangle \rangle$

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Other datatypes and $\tilde{\mu}$

$c ::= \langle t \parallel e \rangle$ command

$t, u ::=$

- | x, y, z
- | $\mu\alpha. c$
- | $\mu(x \cdot \alpha). c$
- | (t, u)

$e, f ::=$

- | \star, α, β
- | $?$
- | $t \cdot e$
- | $\tilde{\mu}(x_1, x_2). c$

Other datatypes and $\tilde{\mu}$

$c ::= \langle t \parallel e \rangle$ command

$t, u ::=$

- | x, y, z
- | $\mu\alpha. c$
- | $\mu(x \cdot \alpha). c$
- | (t, u)

$e, f ::=$

- | \star, α, β
- | $\tilde{\mu}x. c$
- | $t \cdot e$
- | $\tilde{\mu}(x_1, x_2). c$

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| $\mu\alpha. c$
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$e, f ::=$

| \star, α, β
| $\tilde{\mu}x. c$
| $t \cdot e$
| $\tilde{\mu}(x_1, x_2). c$

$\langle \mu\alpha. c \parallel e \rangle \triangleright_{\mu} c[e/\alpha]$
 $\langle t \parallel \tilde{\mu}x. c \rangle \triangleright_{\tilde{\mu}} c[t/x]$
 $\langle (t_1, t_2) \parallel \tilde{\mu}(x_1, x_2). c \rangle \triangleright_{\otimes} c[t_1/x_1, t_2/x_2]$
 $\langle \mu(x \cdot \alpha). c \parallel t \cdot e \rangle \triangleright_{\rightarrow} c[t/x, e/\alpha]$

Other datatypes and $\tilde{\mu}$

$c ::= \langle t \parallel e \rangle$ command

$t, u ::=$

- | x, y, z
- | $\mu\alpha. c$
- | $\mu(x \cdot \alpha). c$
- | (t, u)
- | $\sigma_i t$

$e, f ::=$

- | \star, α, β
- | $\tilde{\mu}x. c$
- | $t \cdot e$
- | $\tilde{\mu}(x_1, x_2). c$

$\langle \mu\alpha. c \parallel e \rangle$	\triangleright_{μ}	$c[e/\alpha]$
$\langle t \parallel \tilde{\mu}x. c \rangle$	$\triangleright_{\tilde{\mu}}$	$c[t/x]$
$\langle (t_1, t_2) \parallel \tilde{\mu}(x_1, x_2). c \rangle$	\triangleright_{\otimes}	$c[t_1/x_1, t_2/x_2]$
$\langle \mu(x \cdot \alpha). c \parallel t \cdot e \rangle$	$\triangleright_{\rightarrow}$	$c[t/x, e/\alpha]$

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- | $\mu(x \cdot \alpha). c$
- | (t, u)
- | $\sigma_i t$

$e, f ::=$

- | \star, α, β
- | $\tilde{\mu}x. c$
- | $t \cdot e$
- | $\tilde{\mu}(x_1, x_2). c$
- | $\tilde{\mu}[(\sigma_1 x_1). c_1 \mid (\sigma_2 x_2). c_2]$

$\langle \mu\alpha. c \parallel e \rangle$	\triangleright_{μ}	$c[e/\alpha]$
$\langle t \parallel \tilde{\mu}x. c \rangle$	$\triangleright_{\tilde{\mu}}$	$c[t/x]$
$\langle (t_1, t_2) \parallel \tilde{\mu}(x_1, x_2). c \rangle$	\triangleright_{\otimes}	$c[t_1/x_1, t_2/x_2]$
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- | $t \cdot e$
- | $\tilde{\mu}(x_1, x_2). c$
- | $\tilde{\mu}[(\sigma_1 x_1). c_1 \mid (\sigma_2 x_2). c_2]$
- | $\pi_i e$

$\langle \mu\alpha. c \parallel e \rangle$	\triangleright_{μ}	$c [e/\alpha]$
$\langle t \parallel \tilde{\mu}x. c \rangle$	$\triangleright_{\tilde{\mu}}$	$c [t/x]$
$\langle (t_1, t_2) \parallel \tilde{\mu}(x_1, x_2). c \rangle$	\triangleright_{\otimes}	$c [t_1/x_1, t_2/x_2]$
$\langle \mu(x \cdot \alpha). c \parallel t \cdot e \rangle$	$\triangleright_{\rightarrow}$	$c [t/x, e/\alpha]$

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$c ::= \langle t \parallel e \rangle$ command

$t, u ::=$

- | x, y, z
- | $\mu\alpha. c$
- | $\mu(x \cdot \alpha). c$
- | (t, u)
- | $\sigma_i t$
- | $\mu[(\pi_1 x_1). c_1 \mid (\pi_2 x_2). c_2]$

$e, f ::=$

- | \star, α, β
- | $\tilde{\mu}x. c$
- | $t \cdot e$
- | $\tilde{\mu}(x_1, x_2). c$
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$\langle \mu\alpha. c \parallel e \rangle$	\triangleright_{μ}	$c[e/\alpha]$
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$\langle \mu(x \cdot \alpha). c \parallel t \cdot e \rangle$	$\triangleright_{\rightarrow}$	$c[t/x, e/\alpha]$

Program equivalences

The η -expansion rules are beautiful and regular.
(Thank you, sequent calculus.)

In the λ -calculus:

$$\begin{array}{l} (t : A \rightarrow B) \triangleright_{\eta} \lambda x. t x \\ \forall C, \quad C[t : A \otimes B] \triangleright_{\eta} \text{let } (x_1, x_2) = t \text{ in } C[(x_1, x_2)] \end{array}$$

In L:

$$\begin{array}{l} (t : A \rightarrow B) \triangleright_{\eta} \mu(x \cdot \alpha). \langle t \parallel x \cdot \alpha \rangle \\ (e : A \otimes B) \triangleright_{\eta} \tilde{\mu}(x_1, x_2). \langle (x_1, x_2) \parallel e \rangle \end{array}$$

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Killer feature!

Type system...

$$\begin{array}{l} t, u ::= x, y, z \mid \mu\alpha. c \mid \mu(x \cdot \alpha). c \mid (t, u) \\ e, f ::= \star, \alpha, \beta \mid \tilde{\mu}x. c \mid t \cdot e \mid \tilde{\mu}(x_1, x_2). c \end{array}$$

$$c ::= \langle t \parallel e \rangle$$

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$$c ::= \langle t \parallel e \rangle \quad \boxed{c : (\Gamma \vdash \Delta)} \quad \boxed{\Gamma \vdash t : A \mid \Delta} \quad \boxed{\Gamma \mid e : A \vdash \Delta}$$

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$$\frac{\Gamma \vdash t : A \mid \Delta \quad \Gamma \mid e : A \vdash \Delta}{\langle t \parallel e \rangle : (\Gamma \vdash \Delta)} \quad \frac{}{\Gamma, x : A \vdash x : A \mid \Delta} \quad \frac{}{\Gamma \mid \alpha : A \vdash \alpha : A, \Delta}$$

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$$\frac{c : (\Gamma, x_1 : A_1, x_2 : A_2 \vdash \Delta)}{\Gamma \mid \tilde{\mu}(x_1, x_2). c : A_1 \otimes A_2 \vdash \Delta} \quad \frac{\Gamma \vdash t_1 : A_1 \mid \Delta \quad \Gamma \vdash t_2 : A_2 \mid \Delta}{\Gamma \vdash (t_1, t_2) : A_1 \otimes A_2 \mid \Delta}$$

Type system...

$$\begin{array}{l} t, u ::= x, y, z \quad | \quad \mu\alpha. c \quad | \quad \mu(x \cdot \alpha). c \quad | \quad (t, u) \\ e, f ::= \star, \alpha, \beta \quad | \quad \tilde{\mu}x. c \quad | \quad t \cdot e \quad | \quad \tilde{\mu}(x_1, x_2). c \end{array}$$

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$$\frac{\Gamma \vdash t : A \mid \Delta \quad \Gamma \mid e : B \vdash \Delta}{\Gamma \mid t \cdot e : A \rightarrow B \vdash \Delta} \quad \frac{c : (\Gamma, x : A \vdash \alpha : B, \Delta)}{\Gamma \vdash \mu(x \cdot \alpha). c : A \rightarrow B \mid \Delta}$$

Type system...

$$\begin{array}{l}
 t, u ::= x, y, z \quad | \quad \mu\alpha. c \quad | \quad \mu(x \cdot \alpha). c \quad | \quad (t, u) \\
 e, f ::= \star, \alpha, \beta \quad | \quad \tilde{\mu}x. c \quad | \quad t \cdot e \quad | \quad \tilde{\mu}(x_1, x_2). c
 \end{array}$$

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$$\frac{c : (\Gamma \vdash \alpha : A, \Delta)}{\Gamma \vdash \mu\alpha. c : A \mid \Delta} \quad \frac{c : (\Gamma, x : A \vdash \Delta)}{\Gamma \mid \tilde{\mu}x. c : A \vdash \Delta}$$

and logic

$$\boxed{\Gamma \vdash t : A \mid \Delta}$$

$$\boxed{\Gamma \mid e : A \vdash \Delta}$$

$$\boxed{c : (\Gamma \vdash \Delta)}$$

Multiple co-variables \longleftrightarrow **classical** logic

call/cc(f) $\stackrel{\text{def}}{:=}$

and logic

$$\boxed{\Gamma \vdash t : A \mid \Delta}$$

$$\boxed{\Gamma \mid e : A \vdash \Delta}$$

$$\boxed{c : (\Gamma \vdash \Delta)}$$

Multiple co-variables \longleftrightarrow **classical** logic

$$\text{call/cc}(f) \stackrel{\text{def}}{=} \mu\alpha. \langle f \parallel (\mu(x \cdot \beta). \langle x \parallel \alpha \rangle) \cdot \alpha \rangle$$

and logic

$$\boxed{\Gamma \vdash t : A \mid \Delta}$$

$$\boxed{\Gamma \mid e : A \vdash \Delta}$$

$$\boxed{c : (\Gamma \vdash \Delta)}$$

Multiple co-variables \longleftrightarrow **classical** logic

$$\text{call/cc}(f) \stackrel{\text{def}}{=} \mu\alpha. \langle f \parallel (\mu(x \cdot \beta). \langle x \parallel \alpha \rangle) \cdot \alpha \rangle$$

Intuitionistic restrictions, either:

- ▷ linear co-context

$$\frac{}{\Gamma, x : A \vdash x : A \mid \emptyset} \qquad \frac{\Gamma \vdash t : A \mid \Delta_1 \quad \Gamma \mid e : B \vdash \Delta_2}{\Gamma \mid t \cdot e : A \rightarrow B \vdash \Delta_1, \Delta_2}$$

- ▷ single variable + shadowing

$$t \ u \stackrel{\text{def}}{=} \mu\alpha. \langle t \parallel u \cdot \alpha \rangle$$

$$\lambda x. t \stackrel{\text{def}}{=} \mu(x \cdot \alpha). \langle t \parallel \alpha \rangle$$

$$t \ u \stackrel{\text{def}}{=} \mu \star. \langle t \parallel u \cdot \star \rangle$$

$$\lambda x. t \stackrel{\text{def}}{=} \mu(x \cdot \star). \langle t \parallel \star \rangle$$

Confluence problem

Computational Classical logic: **hard !**

$$\langle \mu\alpha. c_1 \parallel \tilde{\mu}x. c_2 \rangle$$

Confluence problem

Computational Classical logic: **hard !**

$$\langle \mu\alpha. c_1 \parallel \tilde{\mu}x. c_2 \rangle \triangleright c_2 [\mu\alpha. c_1/x]$$

Confluence problem

Computational Classical logic: **hard !**

$$c_1 [\tilde{\mu}x. c_2/\alpha] \triangleleft \langle \mu\alpha. c_1 \parallel \tilde{\mu}x. c_2 \rangle \triangleright c_2 [\mu\alpha. c_1/x]$$

Confluence problem

Computational Classical logic: **hard !**

$$c_1 [\tilde{\mu}x. c_2/\alpha] \triangleleft \langle \mu\alpha. c_1 \parallel \tilde{\mu}x. c_2 \rangle \triangleright c_2 [\mu\alpha. c_1/x]$$

$$\langle x \parallel \alpha \rangle \triangleleft \langle \mu_{-}. \langle x \parallel \alpha \rangle \parallel \tilde{\mu}_{-}. \langle y \parallel \beta \rangle \rangle \triangleright \langle y \parallel \beta \rangle$$

Confluence problem

Computational Classical logic: **hard !**

$$c_1 [\tilde{\mu}x. c_2/\alpha] \triangleleft \langle \mu\alpha. c_1 \parallel \tilde{\mu}x. c_2 \rangle \triangleright c_2 [\mu\alpha. c_1/x]$$

$$\langle x \parallel \alpha \rangle \triangleleft \langle \mu_{-}. \langle x \parallel \alpha \rangle \parallel \tilde{\mu}_{-}. \langle y \parallel \beta \rangle \rangle \triangleright \langle y \parallel \beta \rangle$$

Arbitrary resolution of the critical pair:

- prefer reducing the term: call-by-value
- prefer reducing the co-term: call-by-name

Confluence problem: values and co-values

$\langle \mu\alpha. c_1 \parallel \tilde{\mu}x. c_2 \rangle$

$t, u ::= \mu\alpha. c \mid x, y, z \mid \mu(x \cdot \alpha). c \mid (t, u)$
 $e, f ::= \tilde{\mu}x. c \mid \star, \alpha, \beta \mid t \cdot e \mid \tilde{\mu}(x_1, x_2). c$

Confluence problem: values and co-values

$$\langle \mu\alpha. c_1 \parallel \tilde{\mu}x. c_2 \rangle$$

$$\begin{array}{l} t, u ::= \mu\alpha. c \quad | \quad x, y, z \quad | \quad \mu(x \cdot \alpha). c \quad | \quad (t, u) \\ e, f ::= \tilde{\mu}x. c \quad | \quad \star, \alpha, \beta \quad | \quad t \cdot e \quad \quad \quad | \quad \tilde{\mu}(x_1, x_2). c \end{array}$$

Prefer reducing the term: $\langle V \parallel \tilde{\mu}x. c \rangle \triangleright_{\tilde{\mu}} c [V/x]$

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Pointers

The duality of computation (Curien and Herbelin, 2000) introduced the syntax (with λ)

Tutorial on computational classical logic and the sequent calculus (Downen and Ariola, 2018) (in JFP)

Examples of PL research it inspired:

- Copatterns: Programming Infinite Structures by Observations
Abel, Pientka, Thibodeau, and Setzer (2013)
- Structures for Structural Recursion
Downen, Johnson-Freyd, and Ariola (2015)
- A classical sequent calculus with dependent types
Miquey (2017)
- Deciding equivalence with sums and the empty type
Scherer (2017)

Section 2

A bit of polarity

Call-by-value, name restrictions: **global** confluence solutions.

$$\begin{array}{l} t, u ::= \mu\alpha. c \quad | \quad V \\ V, W ::= \quad \quad \quad x, y, z \quad | \quad \mu(x \cdot \alpha). c \quad | \quad (t, u) \\ e, f ::= \quad \quad \quad \tilde{\mu}x. c \quad | \quad \star, \alpha, \beta \quad | \quad t \cdot e \quad \quad \quad | \quad \tilde{\mu}(x_1, x_2). c \end{array}$$

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Polarity: a **type-directed** solution.

$$\begin{array}{l} P, Q ::= X^+, Y^+ \quad | \quad P \otimes Q \quad | \quad P \oplus Q \quad | \quad \dots \quad | \quad \langle N \rangle^+ \\ M, N ::= X^-, Y^- \quad | \quad P \rightarrow N \quad | \quad M \times N \quad | \quad \dots \quad | \quad \langle P \rangle^- \end{array}$$

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Negative terms (by-name) are inert values.

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Negative terms (by-name) are inert values.

Positive co-terms (by-value) are inert co-values.

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 S ::= \tilde{\mu}^+x. c \quad | \quad \star, \alpha, \beta \quad \quad \quad | \quad V \cdot S \quad \quad \quad | \quad \tilde{\mu}(x_1, x_2). c
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Polarized reduction

$$\begin{array}{lcl} \langle \mu\alpha. c \parallel e \rangle & \triangleright_{\mu} & c [e/\alpha] \\ \langle t \parallel \tilde{\mu}x. c \rangle & \triangleright_{\tilde{\mu}} & c [t/x] \\ \langle (t_1, t_2) \parallel \tilde{\mu}(x_1, x_2). c \rangle & \triangleright_{\otimes} & c [t_1/x_1, t_2/x_2] \\ \langle \mu(x \cdot \alpha). c \parallel t \cdot e \rangle & \triangleright_{\rightarrow} & c [t/x, e/\alpha] \end{array}$$

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Strong values (Focalization)

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What about (t, u) or $t \cdot e$ when t (for example) is not a value?

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What about (t, u) or $t \cdot e$ when t (for example) is not a value?

Two design options:

- Force the user to decide an evaluation order.

$$\langle t \parallel \tilde{\mu}x. \langle (x, u) \parallel \dots \rangle \rangle \qquad \langle t \parallel \tilde{\mu} \langle \dots \parallel x \cdot e \rangle. \rangle$$

- Add an automatic reduction with arbitrary order: Wadler's ζ -rules.

$$\langle (t, u) \parallel S \rangle^+ \triangleright_{\zeta, \text{pair}.1}^{t \notin V} \langle t \parallel \tilde{\mu}x. \langle (x, u) \parallel S \rangle^+ \rangle$$

Polarity: example

Recall : $t \ u \stackrel{\text{def}}{=} \mu \alpha. \langle t \parallel u \cdot \alpha \rangle^-$

Consider $(\lambda x. t) \ u$:

$$\langle \lambda x. t \parallel u \cdot \star \rangle^-$$

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$\rightsquigarrow u$ is a value,

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$P \rightarrow N$: call-by-value function

$\langle M \rangle^+ \rightarrow N$: call-by-name function

Polarity: summary

We name “L” the polarized $\mu\tilde{\mu}$ calcul $\{us,i\}$.

A **type-directed** resolution of classical non-confluence.

Lets you mix call-by-value and call-by-name (like CBPV).

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Pointers : Girard's LC ('91), Munch-Maccagnoni's PhD thesis (2013)/(LICS'14).

Section 3

Effects & Purity

(Inspired by Paul's nice CBPV examples)

How to add a primitive effect to L?

Head reduction can be seen as a function

$$(\triangleright) : \text{Com} \rightarrow \text{Com}$$

written as a relation: $c_1 \triangleright c_2$ for $(\triangleright c_1) = c_2$.
(normal commands reduce into themselves)

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(normal commands reduce into themselves)

Now pick an effect T as a monad on Set , and extend

$$(\triangleright_T) : \text{Com} \rightarrow T(\text{Com})$$

Printing

Output literals $o \in O$.

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Repeated reductions, Kleisli composition :

$$\text{print}(1). \text{print}(2). c \quad \triangleright_T \quad ([1], \text{print}(2). c) \quad \triangleright_T \quad ([1, 2], c)$$

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Other effects :

- ↪ Non-termination ($TX = 1 + X$, fail),
- ↪ Non-determinism ($TX = \mathcal{P}_{fin}(X)$, flip()).

Effects and polarity: as it should

“(print 1; (x, y))”:

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Negative term : value, will block effects until the function is forced.

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$$\begin{aligned} (t : A \rightarrow B) &\triangleright_{\eta} \mu(x \cdot \alpha). \langle t \parallel x \cdot \alpha \rangle \\ (e : A \otimes B) &\triangleright_{\eta} \tilde{\mu}(x_1, x_2). \langle (x_1, x_2) \parallel e \rangle \end{aligned}$$

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We already had the right η -laws to account for effects.

Purity and substitution

A pure term t can be substituted in any context (referential transparency).
What should be a pure coterms e ?

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↔ Linearity and thunkability

Symmetric notions of purity from **substitution properties**.

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t **thunkable** (“ t is pure”) $\stackrel{\text{def}}{:=}$

$$\forall c[x], \quad c[t/x] \simeq \langle t \parallel \tilde{\mu}x. c \rangle$$

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$$c[t/x] \not\simeq \langle t \parallel \tilde{\mu}x. c \rangle$$

For example:

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For example:

$\text{print}(2). c'$ (non-commutative)

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$$c[t/x] \not\simeq \langle t \parallel \tilde{\mu}x. c \rangle$$

For example:

$$\langle x \parallel \tilde{\mu}x_1. \langle x \parallel \tilde{\mu}x_2. \rangle c' \rangle \quad (\text{duplicating})$$

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$$c[t/x] \not\approx \langle t \parallel \tilde{\mu}x. c \rangle$$

For example:

$$\langle y \parallel \star \rangle \quad (\text{erasing})$$

Thunkable terms

t **thunkable** (“ t is pure”) $\stackrel{\text{def}}{:=}$

$$\forall c[x], \quad c[t/x] \simeq \langle t \parallel \tilde{\mu}x. c \rangle$$

All V are thunkable. Semantic notion of value.

Not thunkable : $t \stackrel{\text{def}}{:=} \mu^+ \alpha. \text{print}(1). \langle x \parallel \alpha \rangle.$

$$c[t/x] \not\simeq \langle t \parallel \tilde{\mu}x. c \rangle$$

For example:

$$\langle y \parallel \star \rangle \quad (\text{erasing})$$

Thunkable : observationally pure commands

$$\mu^+ \alpha. \langle V \parallel \alpha \rangle$$

$$\mu \alpha. \text{flip}(x). \langle x \parallel \tilde{\mu}[(\sigma_1 -). \langle V \parallel \alpha \rangle \mid (\sigma_2 -). \langle V \parallel \alpha \rangle] \rangle$$

Linear co-terms

e **linear** (“ e is strict”) $\stackrel{\text{def}}{:=}$

$$\forall c[\alpha], \quad c[e/\alpha] \simeq \langle \mu\alpha. c \parallel e \rangle$$

All S are linear. Semantic notion of co-value.

Not linear : $e \stackrel{\text{def}}{:=} \tilde{\mu}^{-x}. \langle y \parallel \alpha \rangle$

$$\text{take } c[\alpha] \stackrel{\text{def}}{:=} \text{fail} \quad \rightsquigarrow \quad c[e/\alpha] \not\simeq \langle \mu\alpha. c \parallel e \rangle$$

Purity and polarity

Consider a one-one command

$$c[y, \beta] : (y : A \vdash \beta : B)$$

$c[y, \beta]$ thunkable $\stackrel{\text{def}}{:=} \mu\beta. c[y, \beta]$ thunkable.
 $c[y, \beta]$ linear $\stackrel{\text{def}}{:=} \tilde{\mu}y. c[y, \beta]$ linear.

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All $y : P \vdash \beta : B$ are linear.

Conclusion

L : polarized versions of the $\lambda\mu\tilde{\mu}$ abstract-machine calculi.

Future pointer: our upcoming tutorial!

Claim: a good **syntax** to talk about

- classical computation
- mixed-polarity equivalences
- effects
- purity

Thanks!

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(Non-)Categorical models

One-one commands $P \vdash Q$ form a category.

One-one commands $M \vdash N$ form a category.

General one-one commands $A \vdash B$ do **not** form a category:

composition may be non-associative: $A \xrightarrow{f} P \xrightarrow{g} N \xrightarrow{h} B$

Adjunction(s) models

Two adjunctions?

- $\langle - \rangle^- \dashv \langle - \rangle^+$
- $\langle - \rangle^+ \dashv \langle - \rangle^-$

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$$\frac{\langle P \rangle^- \xrightarrow{\text{lin}} N}{P \xrightarrow{\text{lin}+\text{thunk}} N} \\ \frac{}{P \xrightarrow{\text{thunk}} \langle N \rangle^-}$$

$$\frac{\langle N \rangle^+ \rightarrow P}{N \rightarrow P} \\ \frac{}{N \rightarrow \langle P \rangle^-}$$

The first is between pure commands (both linear and thunkable).
Minimal effects.

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The first is between pure commands (both linear and thunkable).
Minimal effects.

The second is between all commands.
Polarised effects.