Equations: a tool for dependent pattern-matching

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May 15, 2017
Outline

1. Setting and overview
2. Internals
3. Recent and future work
1 Setting and overview

2 Internals

3 Recent and future work
Calculus of Inductive Constructions.

Type families.

\texttt{Coq} provides a simple and direct way to do pattern-matching.

Not easy to program with dependent types.
**Type families**

\[ \text{Inductive vect (A : Type) : nat → Type :=} \]
\[ \mid \text{vnil : vect A 0} \]
\[ \mid \text{vcons (n : nat) : A → vect A n → vect A (S n).} \]

How do you write this?

\[ \text{Definition tail \{A n\} (v : vect A (S n)) : vect A n :=} \]
\[ \text{match v with} \]
\[ \mid \text{vcons _ v’ ⇒ v’} \]
\[ \mid _ ⇒ ??? \]
\[ \text{end.} \]
Type families

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| vnil : vect A 0
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But not

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Main features:

- Function definition through a list of clauses.
- Generation of equations.
- Principle of functional elimination.
- Support for refinement, well-founded recursion.

A few more:

- Tactics for `EQUATIONS` support.
- Replacement for dependent destruction.
- Automatic derivation of various classes about inductive types.
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Splitting tree

Most basic version of the splitting tree:

```plaintext
type context_map = context * pattern list * context

type splitting =
  | Compute of context_map * term
  | Split of context_map * int * splitting option array
```

- **Split** refers to the elimination of one variable.
- **Compute** refers to the right-hand side of a clause.

Build a splitting tree by eliminating variables until it covers every clause provided by the user.
Consider a variable \((x : I \overrightarrow{u})\) in the context.

1. Generalize the variable.

To prove \(P\), it is enough to prove:

\[
\forall \overrightarrow{v} \ (y : I \overrightarrow{v}), (\overrightarrow{u}; x) = (\overrightarrow{v}; y) \rightarrow P
\]
Eliminating a variable

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3. In each branch of the inductive type, simplify the generated equalities.

For instance, in \texttt{tail} we would simplify equalities like:

\[
(S\ n; \ v) = (0; \ vnil) \\
(S\ n; \ v) = (S\ n'; \ vcons\ x\ v')
\]
Simplification steps

At each step, there is an equality \( t = u \) at the head of the goal. We want to unify \( t \) and \( u \). Five possible steps:
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2. **Solution:** $x = t$ where $x$ is a variable

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4. **Conflict:** \( C \vec{u} = D \vec{v} \)

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Solve the goal immediately.
Since we cannot modify pattern-matching in itself, we need to produce a term that will be accepted by Coq to witness the context and goal changes related to these simplification steps.

- Using tactics: easy enough to implement, but risk of coherence problems.
- Writing ”manually” the terms: more work to implement, but precise.
Equations unzip \{A B n\} (v : vect (A * B) n) : vect A n * vect B n :=
unzip vnil := (vnil, vnil);
unzip (vcons (pair a b) v) ⇐ unzip v ⇒ {
| pair v w ⇒ (vcons a v, vcons b w)
}.

In this case, the right-hand side is not a Compute node. Instead we:

- typecheck the term unzip v under the current (and possibly refined) context;
- add a pattern in the current context_map;
- process the rest of this node to produce an auxiliary definition;
- apply this auxiliary definition to the current variables and the term which is refined.
A word about refinement

\[
\text{Equations unzip } \{A \ B \ n\} (v : \text{vect (}A \ B\text{) }n) : \text{vect } A \ n \ \ast \ \text{vect } B \ n \ :=
\text{unzip vnil := (vnil, vnil)} ;
\text{unzip (vcons (pair } a \ b\text{) }v) \Leftarrow \text{unzip } v \Rightarrow \{
\text{unzip (vcons (pair } a \ b\text{) }_\text{) (pair } v \ w\text{) } \Rightarrow (\text{vcons } a \ v, \text{vcons } b \ w)
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Local definition (where keyword)

- Similar to a let-in.
- Provide a definition through a splitting tree, as usual.
- Possible to combine it with well-founded recursion to obtain nested or mutual recursion.
Proof irrelevance was used to prove the fixpoint lemmas about well-founded recursion. We avoid it by proving it directly for the accessibility relation. Additionally, a lot of work about the axiom K...
When we generalize a variable \((x : I \bar{u})\), we introduce equalities. Before, we used heterogeneous equalities where needed.

- Easy to manipulate (less dependency between equalities).
- Entails the use of the axiom K.

Now we use homogeneous equalities between telescopes.

- Have to be careful because each equality depends on the previous ones.
- The use of the rule K is targeted to a specific type.
Pattern-matching in CoQ can do part of our work to make terms look nicer.

```
match x as x' in I u' return P u' x' with
| C y ⇒ ...
| D z ⇒ ...
end
```

In each branch, $u'$ and $x'$ are instantiated with the actual indices and constructor. This corresponds to a solution step.

- Try to solve as many solution steps as possible through this mechanism.
- Might need to introduce cuts to compensate.
Pattern-matching in Coq can do part of our work to make terms look nicer.

```coq
match x as x' in I u' return forall (a : T u'), P u' x' with
  | C y ⇒ ...
  | D z ⇒ ...
end a
```

In each branch, \(u'\) and \(x'\) are instantiated with the actual indices and constructor. This corresponds to a solution step.

- Try to solve as many solution steps as possible through this mechanism.
- Might need to introduce cuts to compensate.
Replacing **FUNCTION**?

- Proving the correctness of the functional graph. For now we only need `forall x, f_ind x (f x)` to derive the functional elimination principle. **FUNCTION** also proves `forall x y, f_ind x y → y = f x`.

- Tracking default cases. **EQUATIONS** fully expands its splitting tree, and therefore loses track of clauses that would cover several constructors at once.

- Keeping the same surface syntax. Ideally, the user would not see any change of any code written with **FUNCTION**, only the underlying code would branch out to **EQUATIONS**.
**Conclusion**

**Equations** was already used successfully for a few applications:

- Normalization of LF.
- Consistency of predicative System F.
- Reflexive tactic to decide equality of polynomials.

**Equations** is available on GitHub[^1] and OPAM. It is still in an experimental state, and not all features discussed here are available in 8.6.

[^1]: [https://github.com/mattam82/coq-equations](https://github.com/mattam82/coq-equations)