A Functional Synchronous Language with Integer Clocks

Adrien Guatto

ENS - PARKAS

Gallium Seminar

Reactive Systems

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- Critical Control Software:

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  - Process unbounded sequences of data
  - ... within bounded memory
  - ... and bounded reaction time.

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- *State machine*-like implementations; how to program them?

Functional Synchronous Programming

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In practice, compilers generate transition functions looking like

```
void f_step(struct f_state *self, int in, int *out);
```

Typically: only assignments, conditionals and function calls in `f_step`

Synchrony and Performance-Sensitive Code

- Traditional use cases: control laws, protocols, etc.
- Signal processing: involve...
  - subtle space/time tradeoffs
  - architecture-dependent optimizations
- Can we use Synchronous Languages for such applications?

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My Long-Term Objective

Design and implement a...

- synchronous functional language
- compiling to hardware and software
- with the usual safety guarantees
- but generating code of a different shape

Ingredients

Integer Clocks

- Compute streams by bursts of value
- Generate array-oriented code from purely functional source

Local Time Scales

- Time may pass faster inside than outside
- Compile to counted loops
- Make the type system more uniform

Linear Higher-Order Functions

- Call every function you receive exactly once
- Enable *modular* compilation to hardware

This Talk

- Present Integer Clocks and Local Time Scales intuitively
  - Reason purely on stream functions à la Lustre, Lucid S., Lucy-n
  - Focus on first-order parts
- Show how the intuitions can be implemented as a type system
  - (Check buffers sizes)
  - Reject non-causal programs
- Discuss soundness results
  - Proof by *realizability*

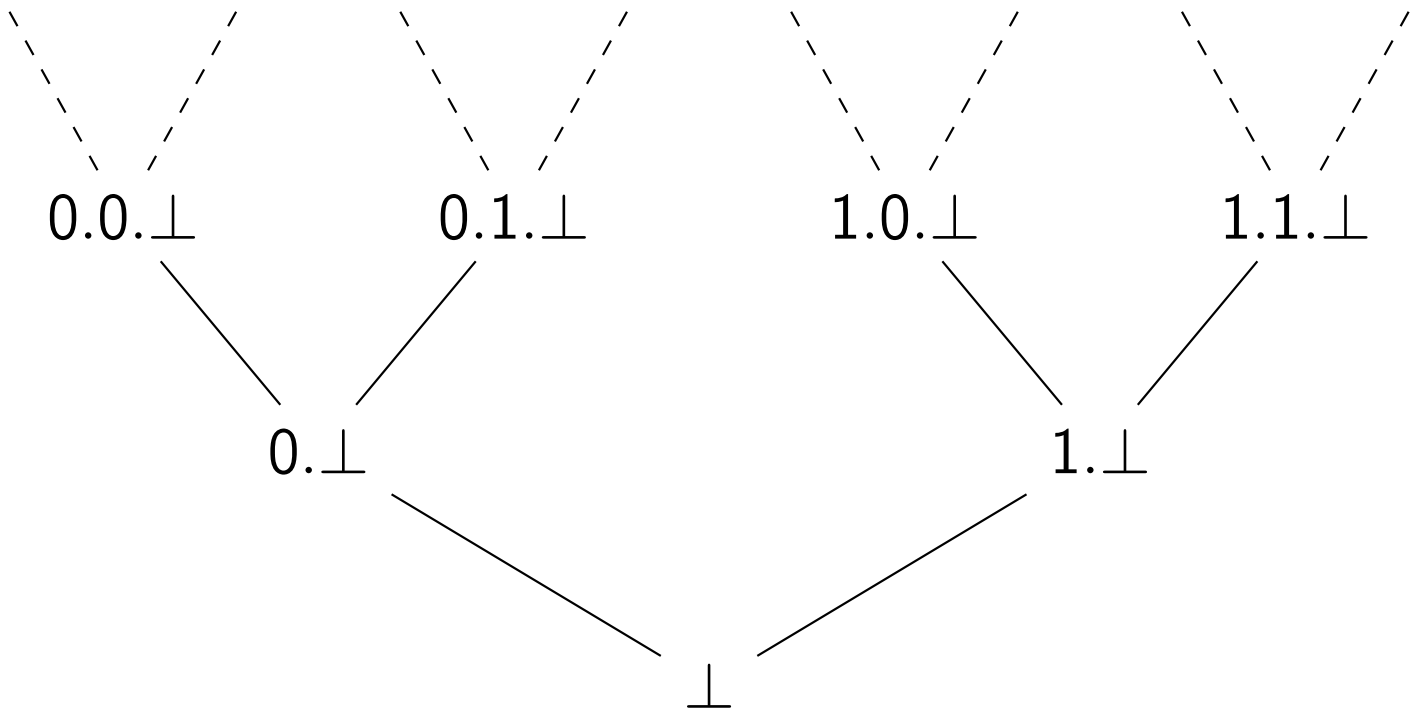
Outline

Streams and Partiality

- Streams are *infinite* sequences of values
  - Think of them as produced by programs running forever
- However, streams may be *partial*, i.e. block after some time!
  - Happens when the producer program does an infinite, silent loop.

Streams and Partiality

- Streams are *infinite* sequences of values
  - Think of them as produced by programs running forever
- However, streams may be *partial*, i.e. block after some time!
  - Happens when the producer program does an infinite, silent loop.
- Here is a picture of $\mathit{Stream}(\mathbb{B})$, ordered by *information*:



Stream Functions (1/2)

Consider the following function

$$\begin{array}{lcl} f & : & \mathit{Stream}(\mathbb{N}) \to \mathit{Stream}(\mathbb{N}) \\ f\,(x.xs) & = & (x+1).(f\ xs) \end{array}$$

Can it be implemented as a state machine?

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Can it be implemented as a state machine? Yes. For example:

$$\begin{array}{lcl} m & : & \mathcal{M}(\mathbb{N},\mathbb{N}) \\ m & = & (\{*\}, *, \lambda(*,x).(*,x+1)) \end{array}$$

The machine m processes one element per transition.
It was easy since the function is *length-preserving*.

Stream Functions (2/2)

What about the following function?

$$\begin{array}{lcl} g & : & \mathit{Stream}(\mathbb{N}) \to \mathit{Stream}(\mathbb{N}) \\ g(x.xs) & = & (x+1).(x-1).(g\ xs) \end{array}$$

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Yes, if we cheat a bit.

$$\begin{array}{lcl} m_1 & : & \mathcal{M}(\mathbb{N}, \mathit{List}(\mathbb{N})) \\ m_1 & = & (\{*\}, *, \lambda(*, x).(*, [x+1; x-1])) \end{array}$$

Another possibility:

$$\begin{array}{lcl} m_2 & : & \mathcal{M}(\mathit{List}(\mathbb{N}), \mathbb{N}) \\ m_2 & = & (\mathbb{N} \cup \{*\}, *, \\ & & \quad \lambda(s, x).\text{if } s = * \text{ then } (\mathit{hd}\ x, \mathit{hd}\ x + 1) \text{ else } (*, s-1)) \end{array}$$

Stream Functions and Clocks

Naively speaking, the function g is not length-preserving.

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However, we can make it so by changing its (co)domain!

$$\begin{array}{lcl} g_1 & : & \mathit{Stream}(\mathit{List}(\mathbb{N})) \to \mathit{Stream}(\mathit{List}(\mathbb{N})) \\ g_1\ ([x].xs) & = & [x+1; x-1].(g_1\ xs) \end{array}$$

$$\begin{array}{lcl} g_2 & : & \mathit{Stream}(\mathit{List}(\mathbb{N})) \to \mathit{Stream}(\mathit{List}(\mathbb{N})) \\ g_2\ ([x].xs) & = & [x+1].(\text{let } [].xs' = xs \text{ in} \\ & & \qquad [x-1].(g_2\ xs')) \end{array}$$

Functions g_1 and g_2 are length-preserving.

Synchronizing Functions

How to describe the relationship between g, g_1 and g_2?

$$
\begin{array}{lcccc}
g & : & \mathit{Stream}(\mathbb{N}) & \rightarrow & \mathit{Stream}(\mathbb{N}) \\
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Remember that g_1 and g_2 work only for specific list sizes:

	Input list sizes	Output list sizes
g_1	$(1)^\omega$	$(2)^\omega$
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These integer streams, *clocks*, fully characterize g_1 and g_2.
We write:

$$\begin{array}{lcc} g_1 & :: & (1) \multimap (2) \\ g_2 & :: & (1\ 0) \multimap (1) \end{array}$$

From Streams to Clocked Streams, and back

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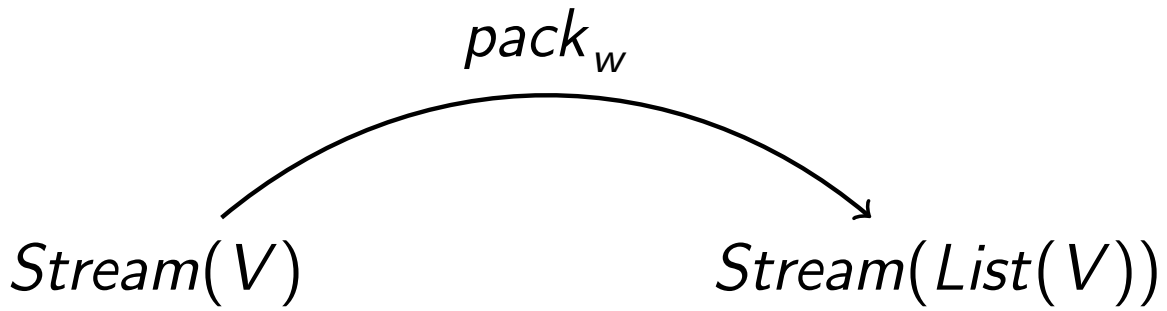

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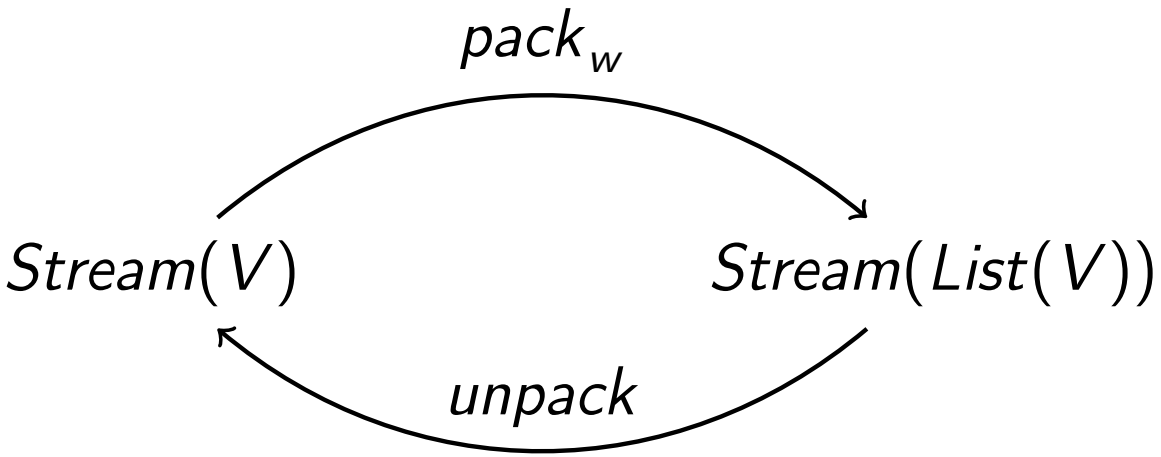

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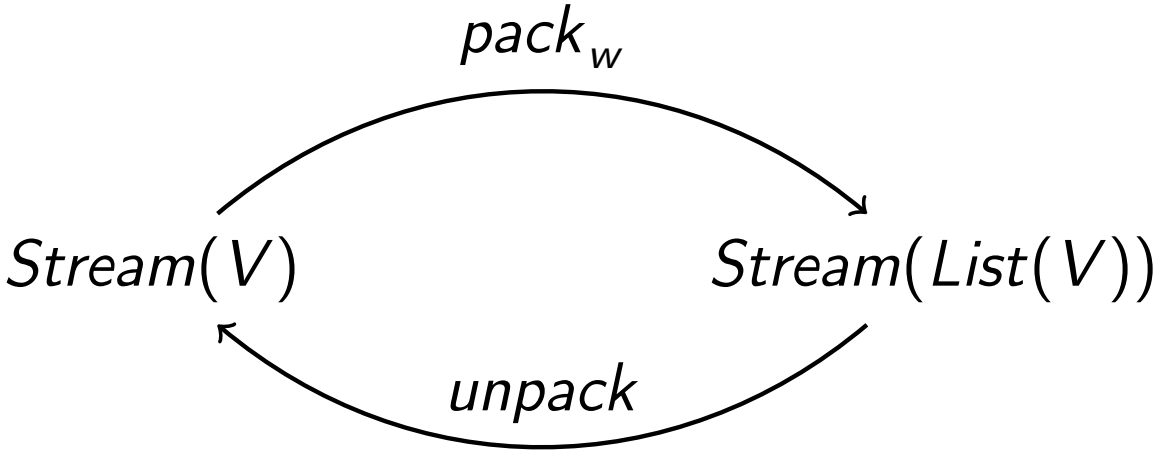

For example:

$x = pack_{1(10)^\omega}\ (a.b.c.d\ldots)$	$[a]$	$[b]$	$[]$	$[c]$	$[]$	$\ldots$
$y = pack_{(02)^\omega}\ (a.b.c.d\ldots)$	$[]$	$[a;b]$	$[]$	$[c;d]$	$[]$	$\ldots$

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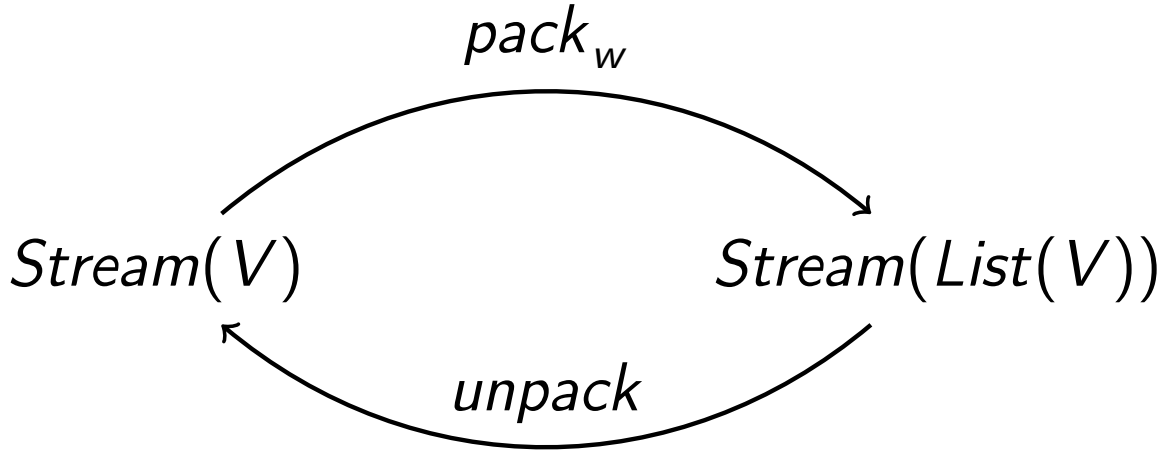

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Obviously:

$$unpack\ x = unpack\ y$$

Synchronous Stream Functions

We now *define* the functions g_1 and g_2 purely from their clocks:

$$\begin{aligned} g_1 &:: (1) \multimap (2) \\ g_1 &= pack_{(2)} \circ g \circ unpack \\ g_2 &:: (1\,0) \multimap (1) \\ g_2 &= pack_{(1)} \circ g \circ unpack \end{aligned}$$

What about the following signature?

$$\begin{aligned} g_3 &::_{?} (0\,1) \multimap (1) \\ g_3 &= pack_{(1)} \circ g \circ unpack \end{aligned}$$

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It is unsound: g breaks this contract at the first time step:

$$g_3\ ([].\bot) \quad = \quad \bot$$

Playing with Synchronous Functions: Buffers (1/3)

$x :: (1\,0)$	$[a]$	$[]$	$[b]$	$[]$	$[c]$	$[]$	$\ldots$
$x' :: (0\,1)$	$[]$	$[a]$	$[]$	$[b]$	$[]$	$[c]$	$\ldots$

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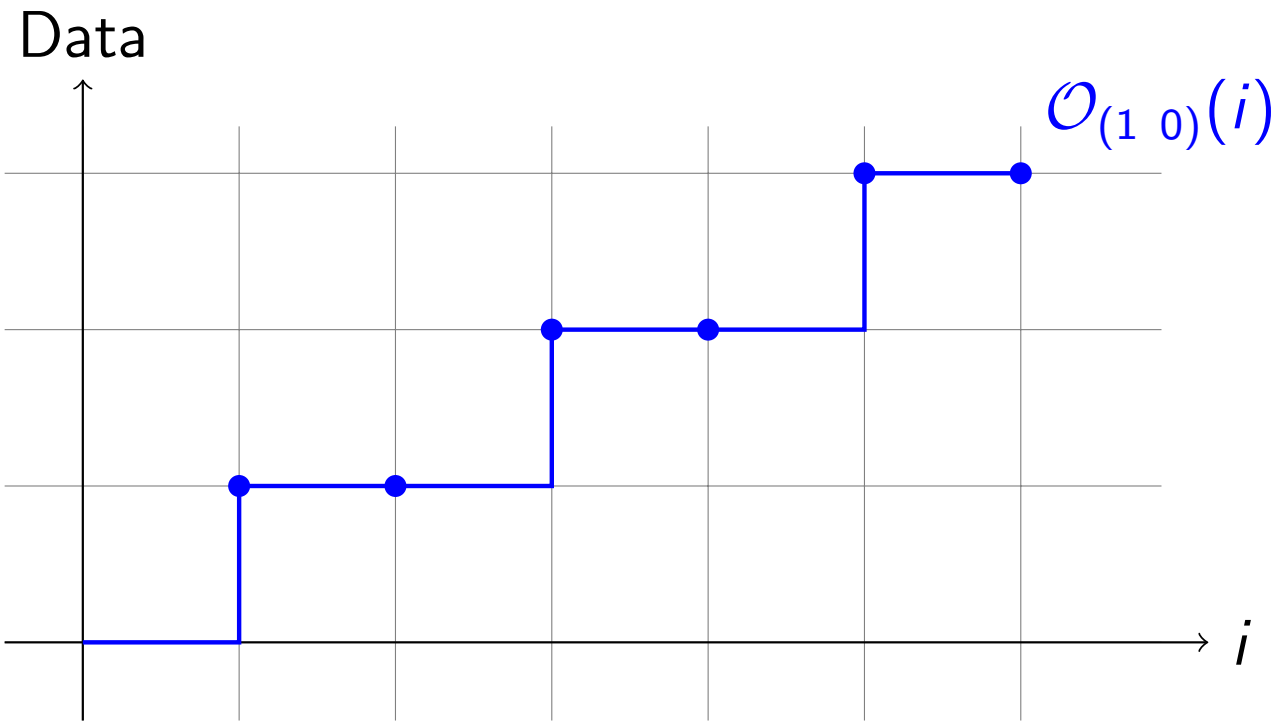

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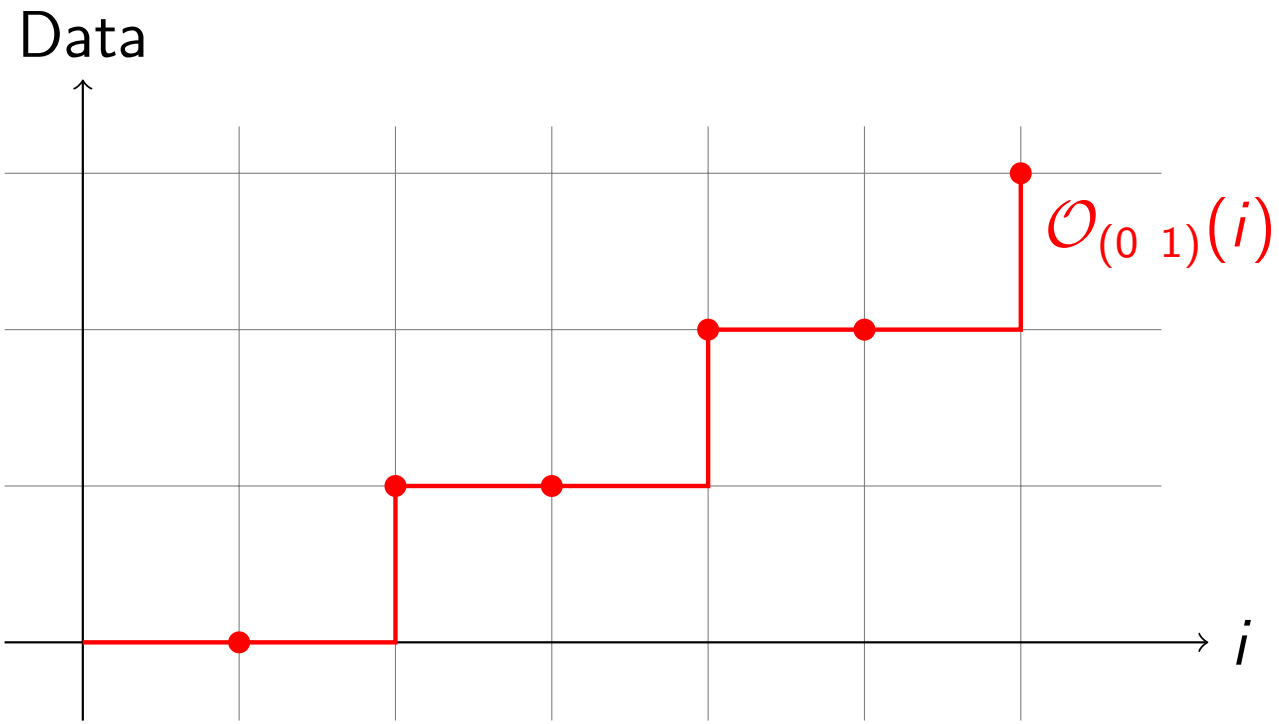

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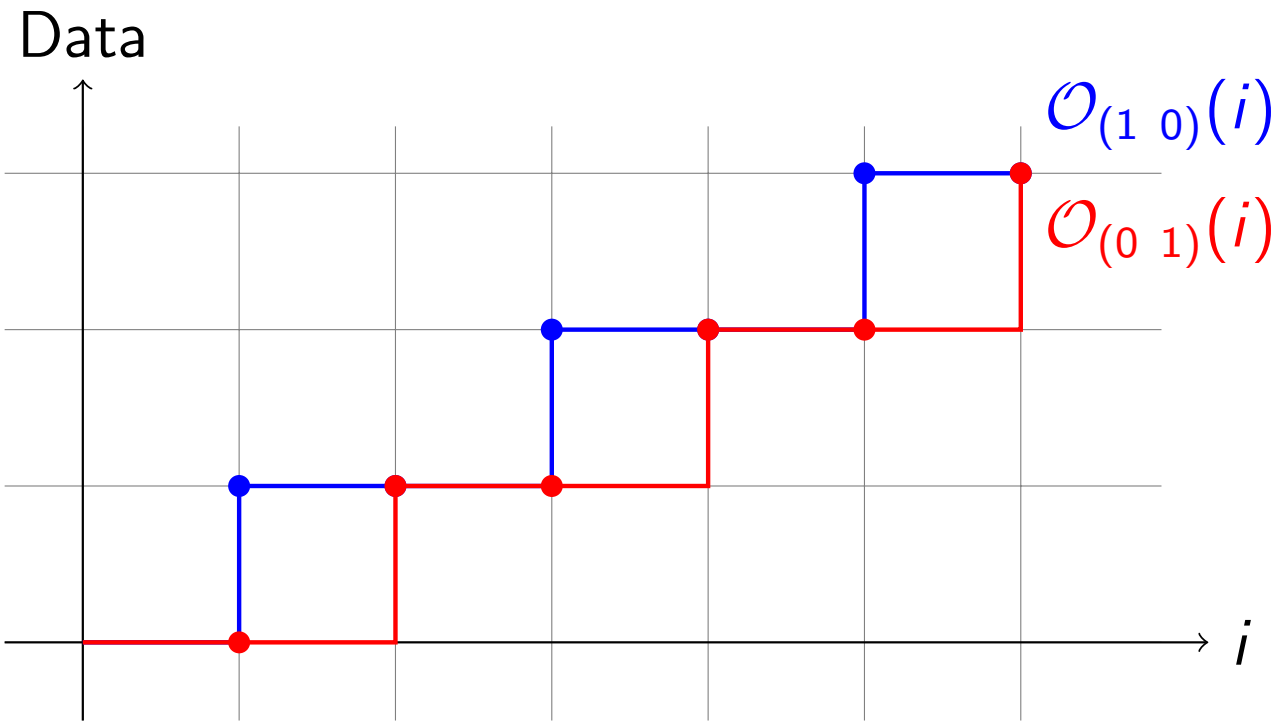

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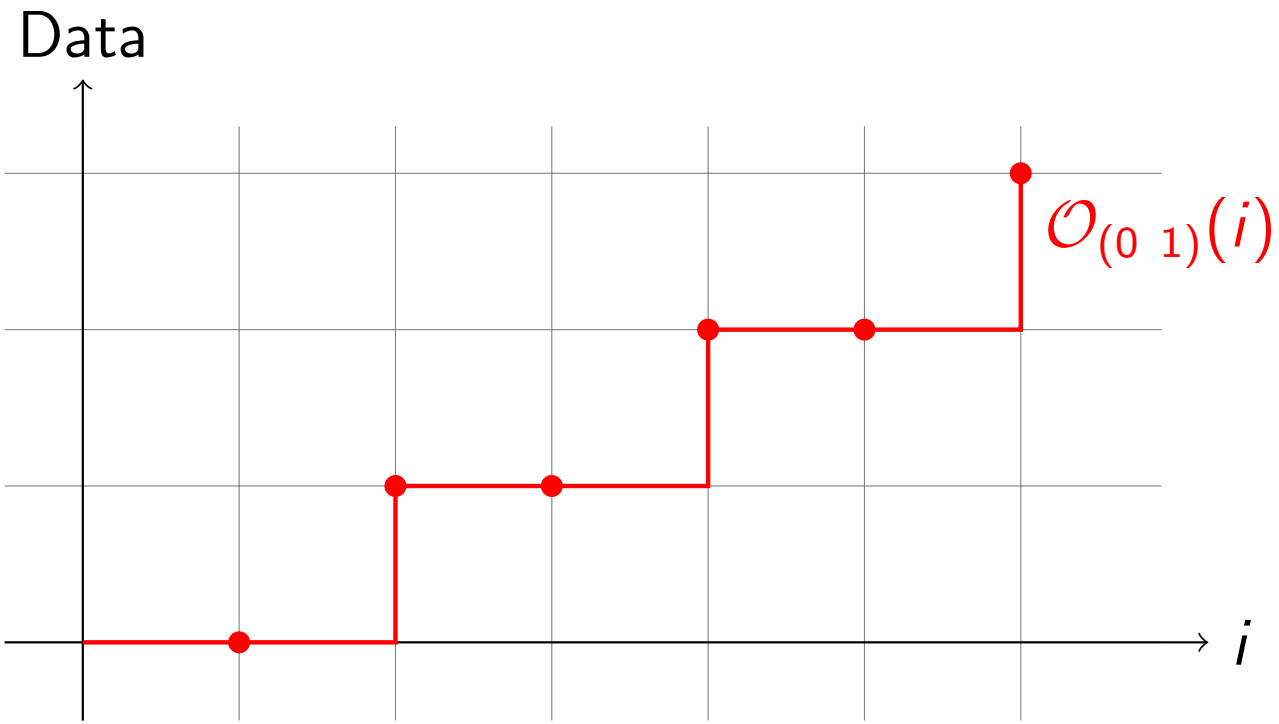

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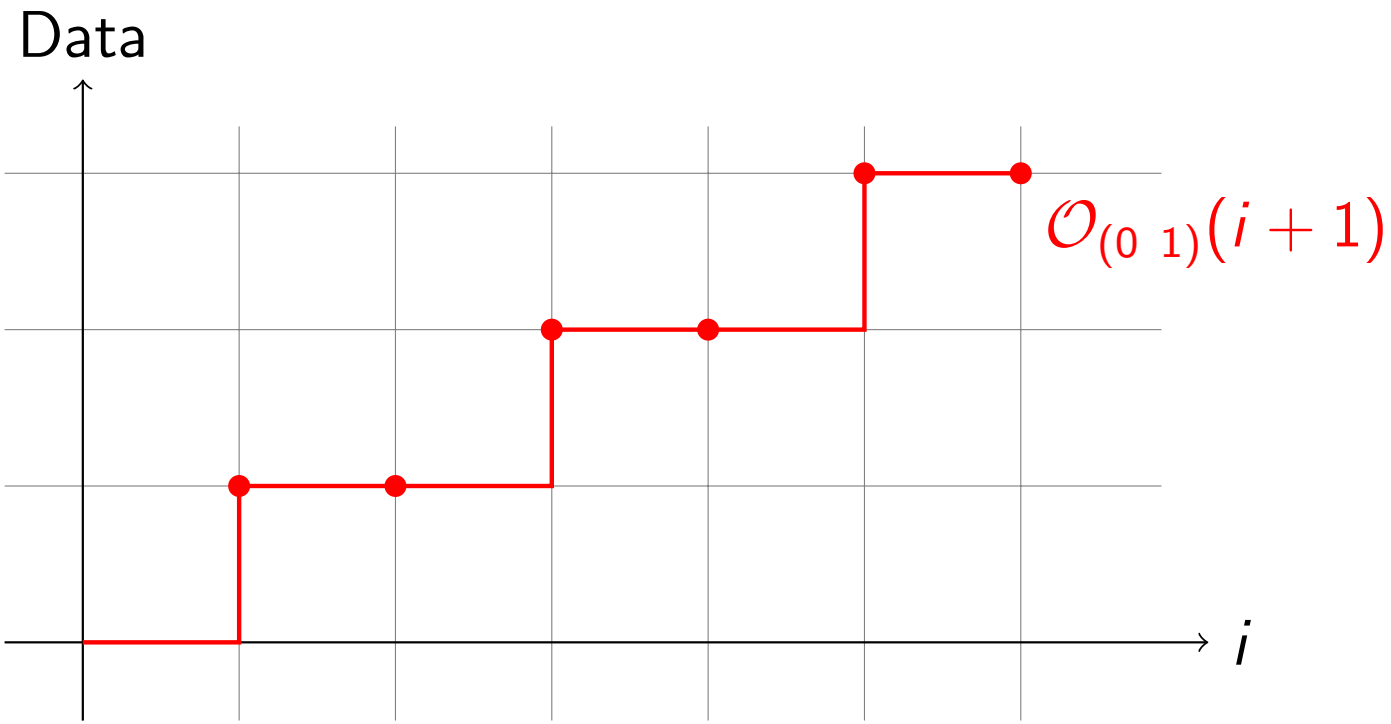

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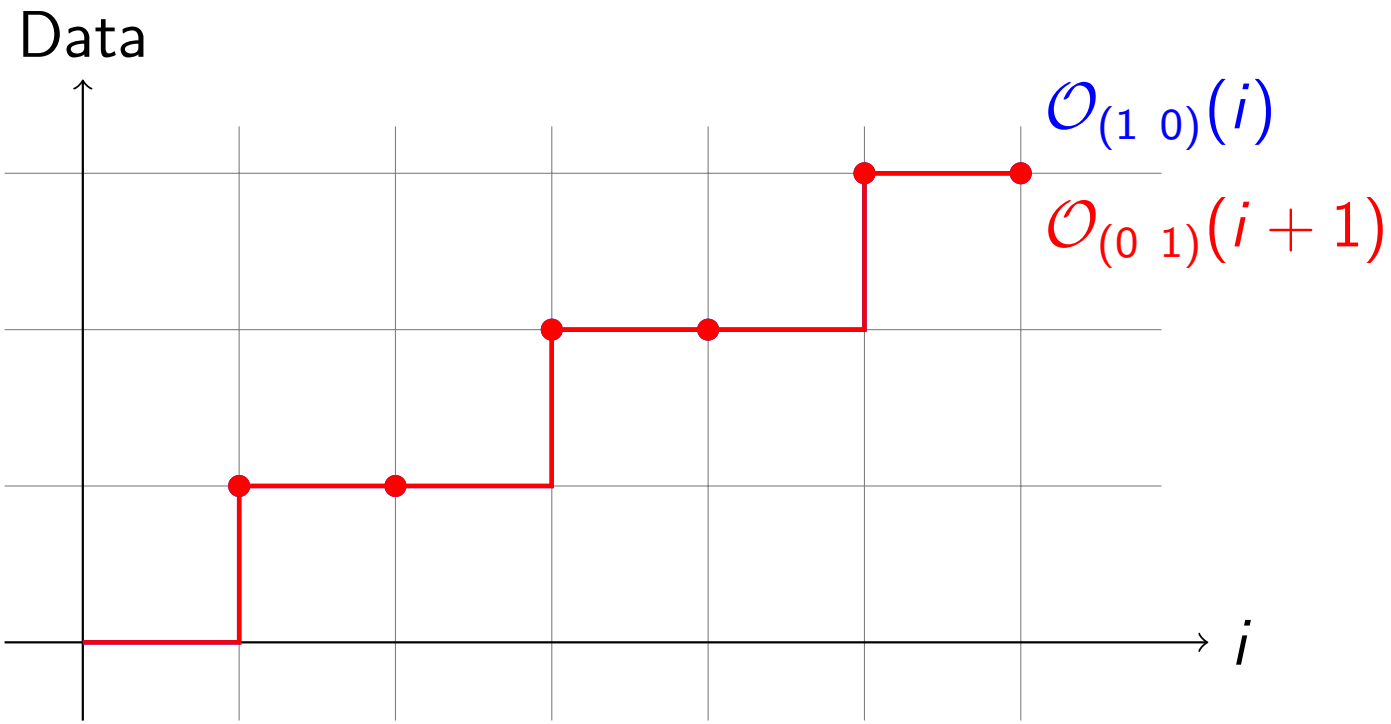

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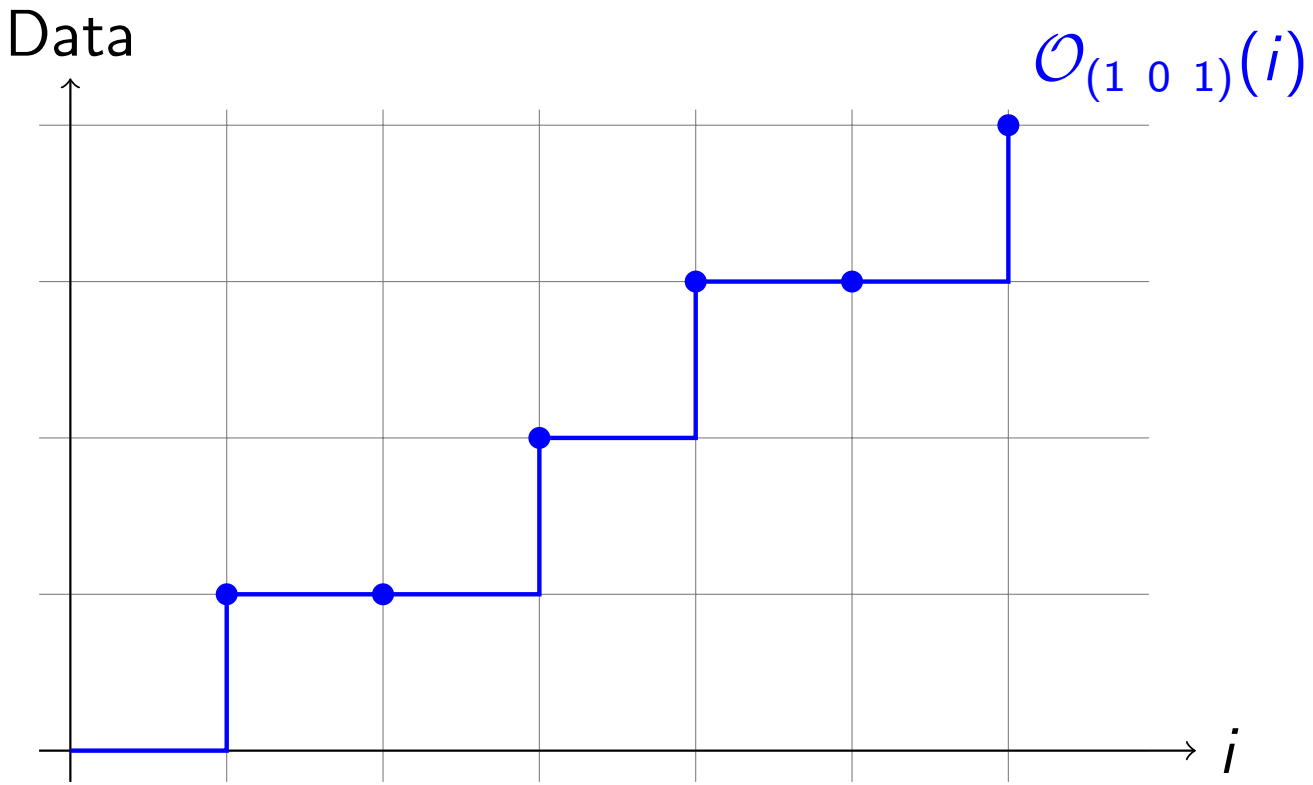

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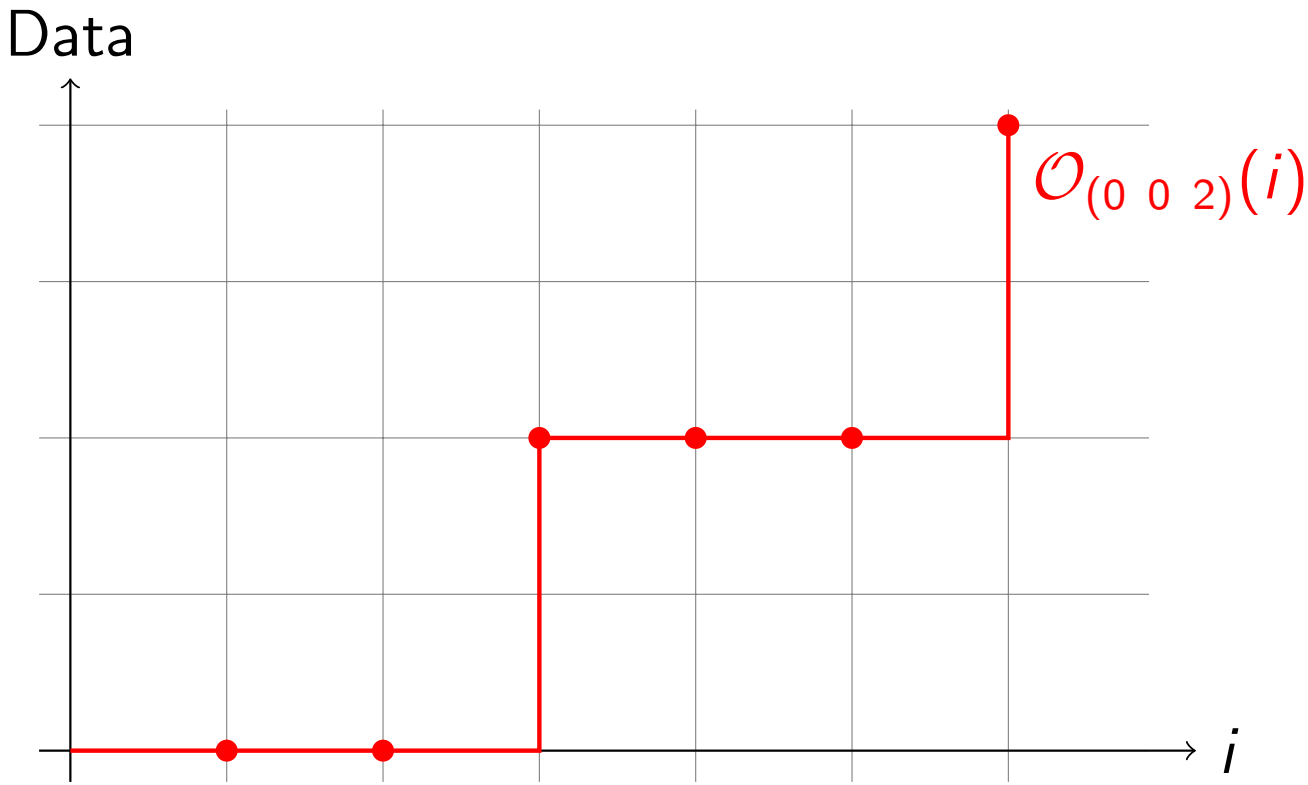

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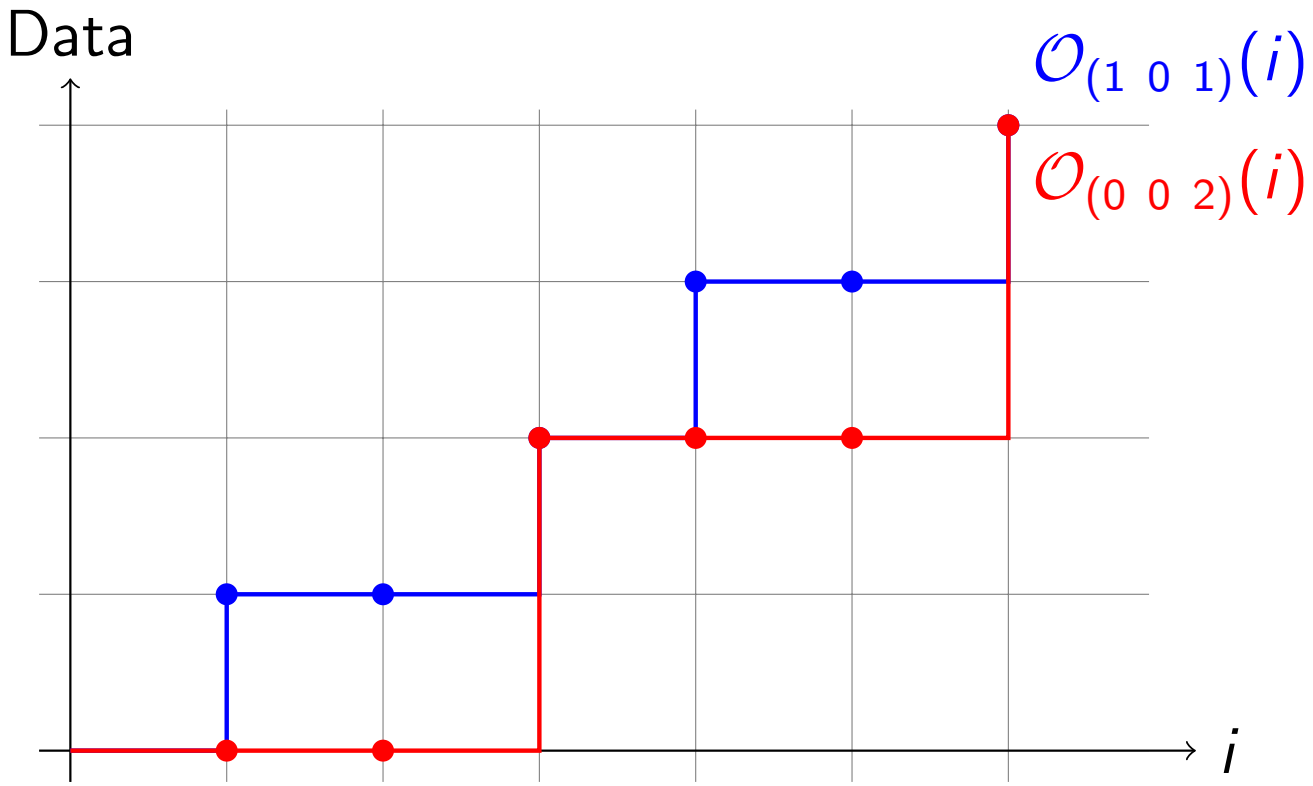

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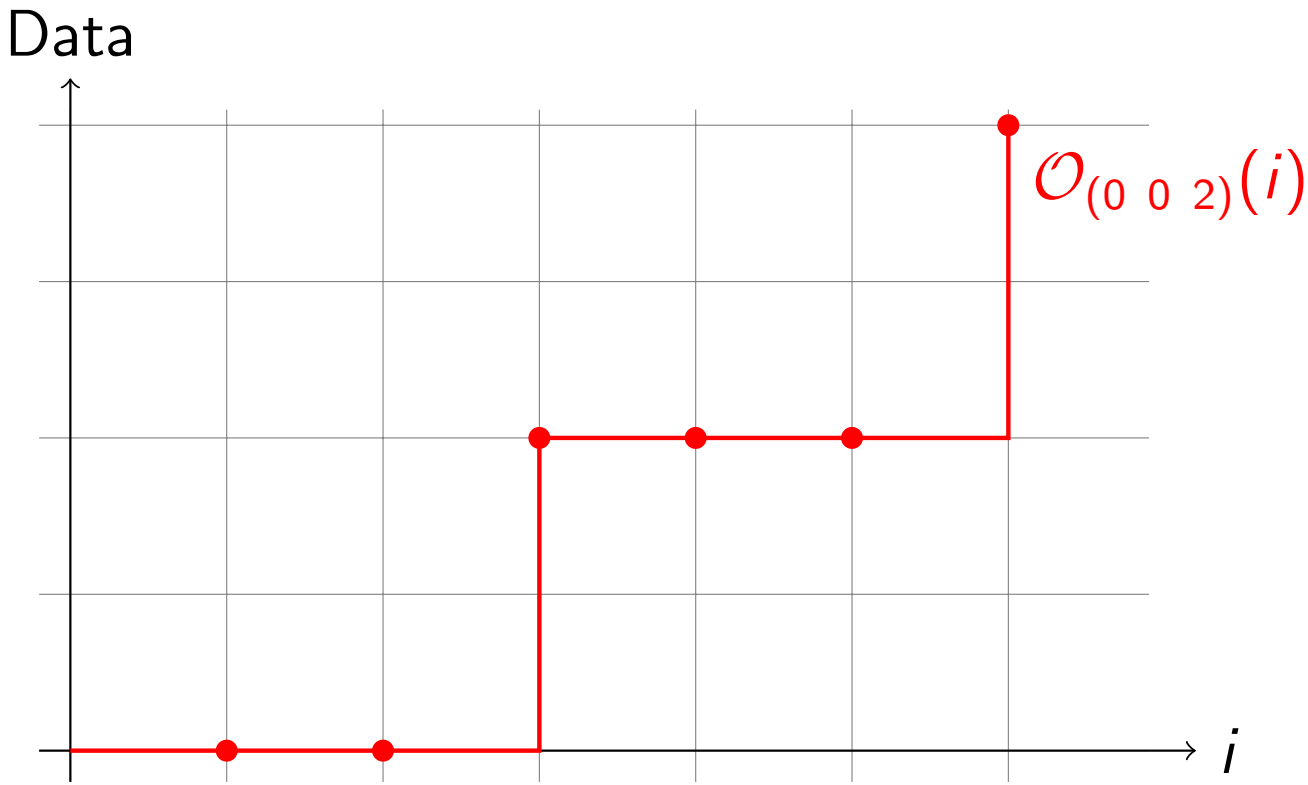

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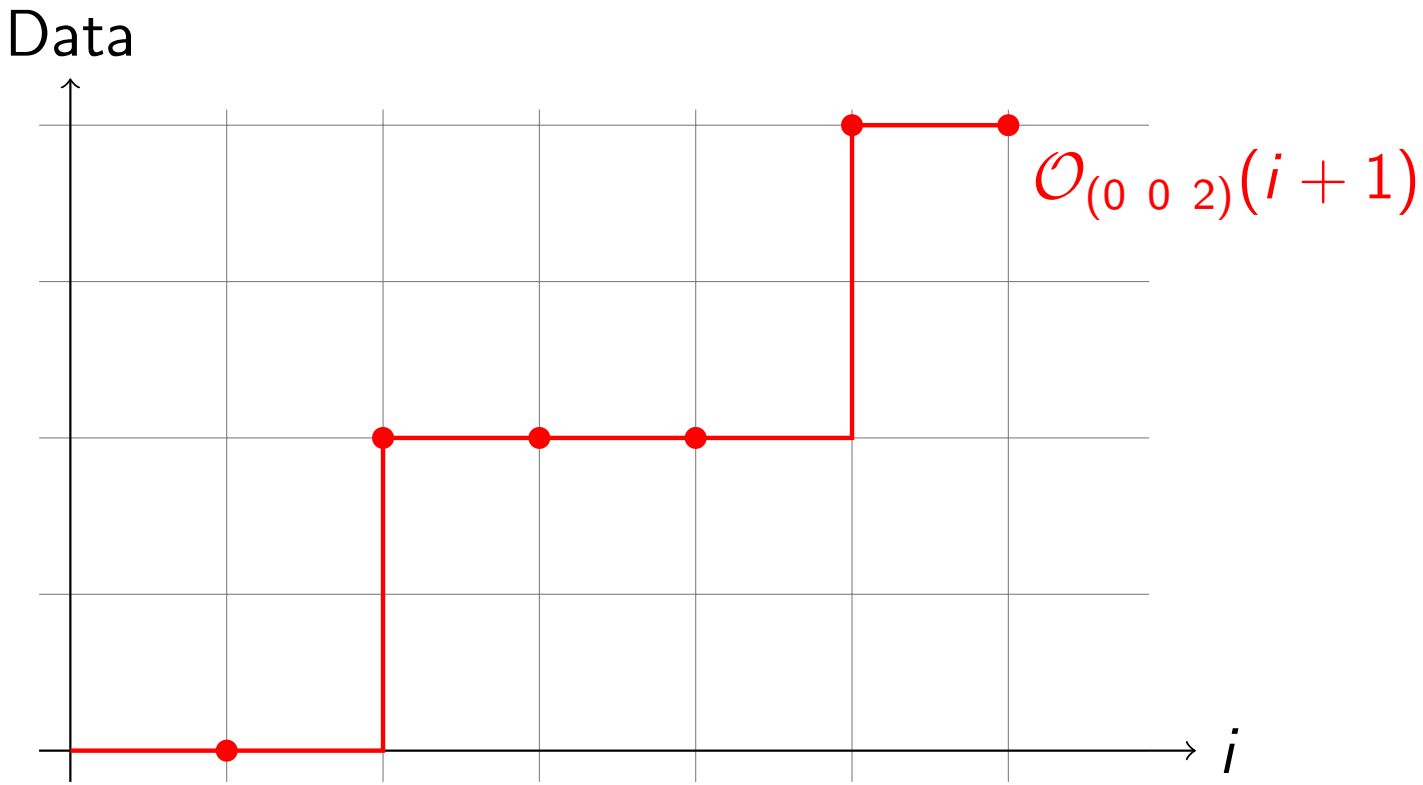

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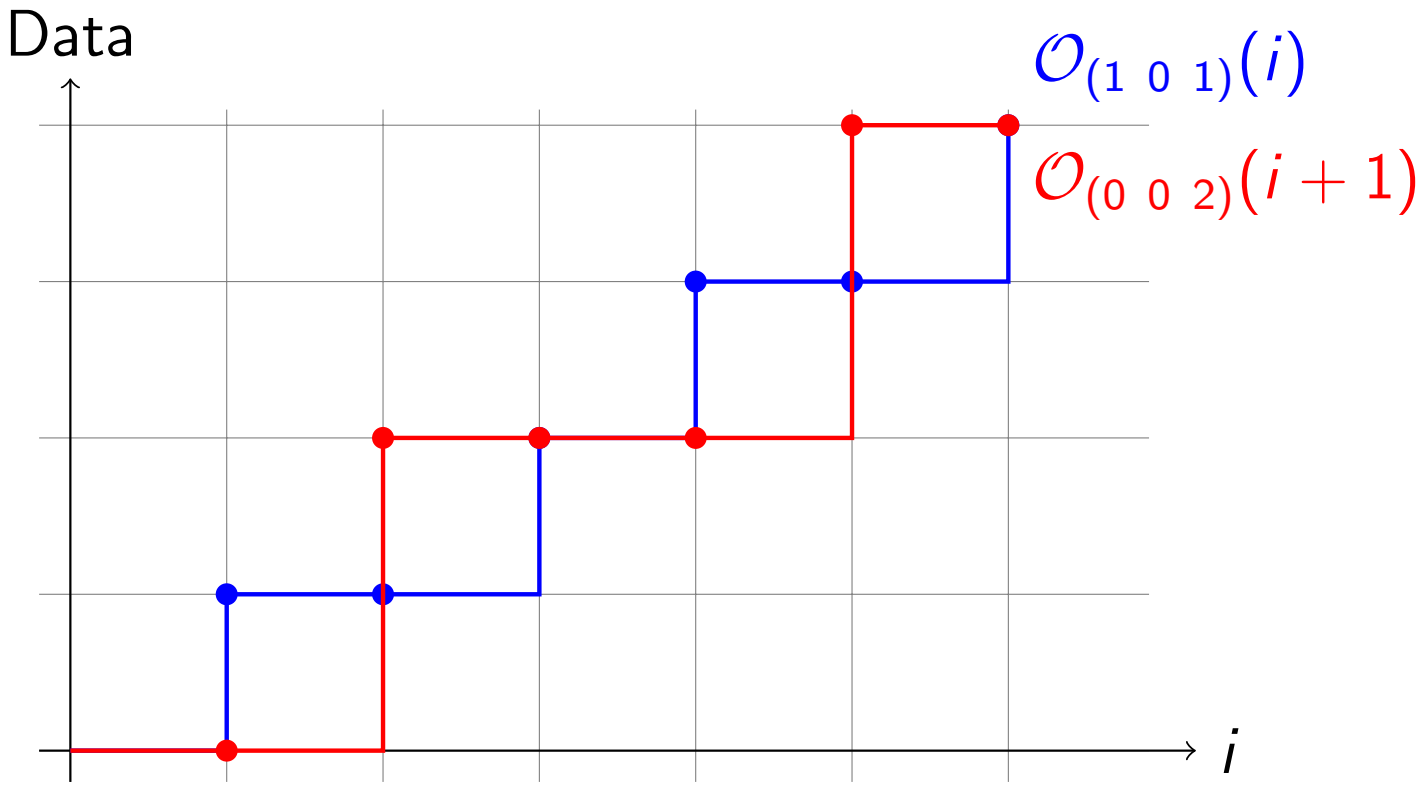

not $(1\ 0\ 1) <:_1 (0\ 0\ 2)$

Playing with Synchronous Functions: Buffers (3/3)

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$$\begin{array}{lcl} h' & :: & w_1 \multimap w_2' \\ h' & = & \mathit{buffer}_{w_2 <:_k w_2'} \circ h \end{array}$$

For example:

$$(1) \multimap (1\,0) <: (1) \multimap (0\,1)$$

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  For example:

$$(0\,1) \multimap (1) <: (1\,0) \multimap (1)$$

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KO
OK
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Given a function $h :: w_1 \multimap w_2$, is it safe to compute $x = h\ x$?
What about...

$$
\begin{array}{lcccc}
h_1 & :: & (1) & \multimap & (1) \\
h_2 & :: & (0\,1) & \multimap & (1\,0) \\
h_3 & :: & (0\,1\,1) & \multimap & (1\,0\,1)
\end{array}
$$

KO
OK
KO

We allow feedback only when $w_2 <:_1 w_1$.
This makes sure that $x = h\ x$ is total.

Playing with Synchronous Functions: Local Time

Take any function f implemented by state machine m, with

$$f :: (1\,0) \multimap (0\,1)$$

We can transform f into f' such that

$$f' :: (1) \multimap (1)$$

What would be m', the implementation of f'?

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What would be m', the implementation of f'?

- A single transition of m' performs two transitions of m
- We write

$$(1\,0) \multimap (0\,1) \uparrow_{(2)} (1) \multimap (1)$$

Local Time Scales and Scatter/Gather

A local time scale comes with a clock w driving its internal time

- E.g. (2 1) begins with two internal steps for one external, etc.

How does the inside sees the outside? The converse?

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- $w_1 \multimap w_2 \uparrow_w w_1' \multimap w_2'$: leaving local time

$$(1\,0\,1) \multimap (0\,1\,1) \quad \uparrow_{(2\,1)} \quad (1) \multimap (1) \qquad \text{OK}$$
$$(0\,1\,1) \multimap (1\,0\,1) \quad \uparrow_{(2\,1)} \quad (1) \multimap (1) \qquad \text{OK}$$

Local Time Scales and Scatter/Gather

A local time scale comes with a clock w driving its internal time

- E.g. $(2\,1)$ begins with two internal steps for one external, etc.

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- $w_1 \multimap w_2 \downarrow_w w_1' \multimap w_2'$: entering local time

$$(1) \multimap (1) \quad \downarrow_{(2\,1)} \quad (1\,0\,1) \multimap (0\,1\,1) \qquad \text{OK}$$
$$(1) \multimap (1) \quad \downarrow_{(2\,1)} \quad (0\,1\,1) \multimap (1\,0\,1) \qquad \text{KO}$$

Scatter/Gather: Streams

Consider two simple examples:

$$(1\,0) \uparrow_{(2)} (1)$$

What is the action of (2) on $(1\,0)$ that gives (1)?

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We can now define:

$$w_1 \uparrow_w w_2 \Leftrightarrow w\ \mathit{on}\ w_1 = w_2$$

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Similarly, $(1) \downarrow_{(2)} (0\,1)$ because $(1) = (2)\ \mathit{on}\ (1\,0)$

Scatter/Gather: Functions

Going back to our first example: $(1\,0) \multimap (0\,1) \uparrow_{(2)} (1) \multimap (1)$. Why?

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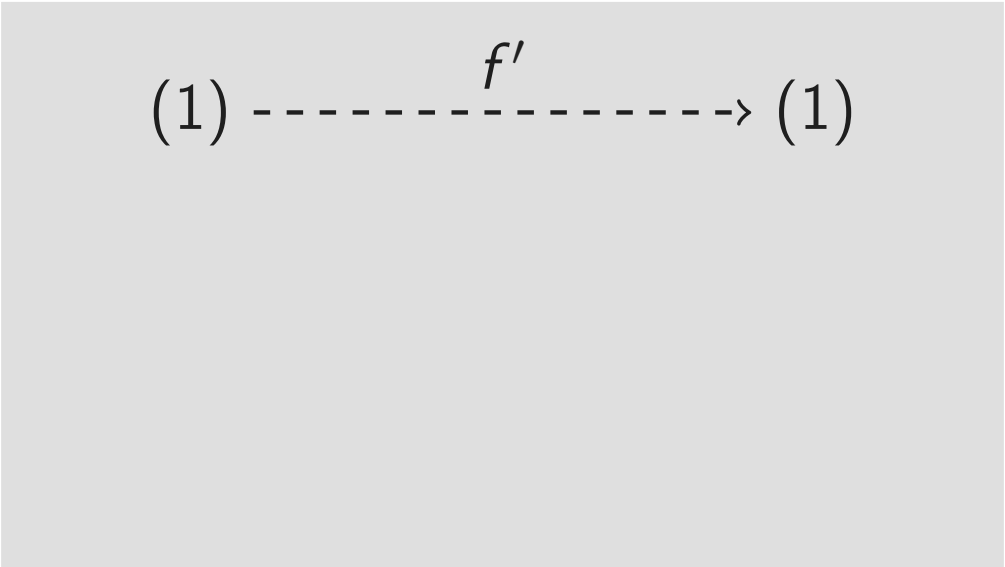

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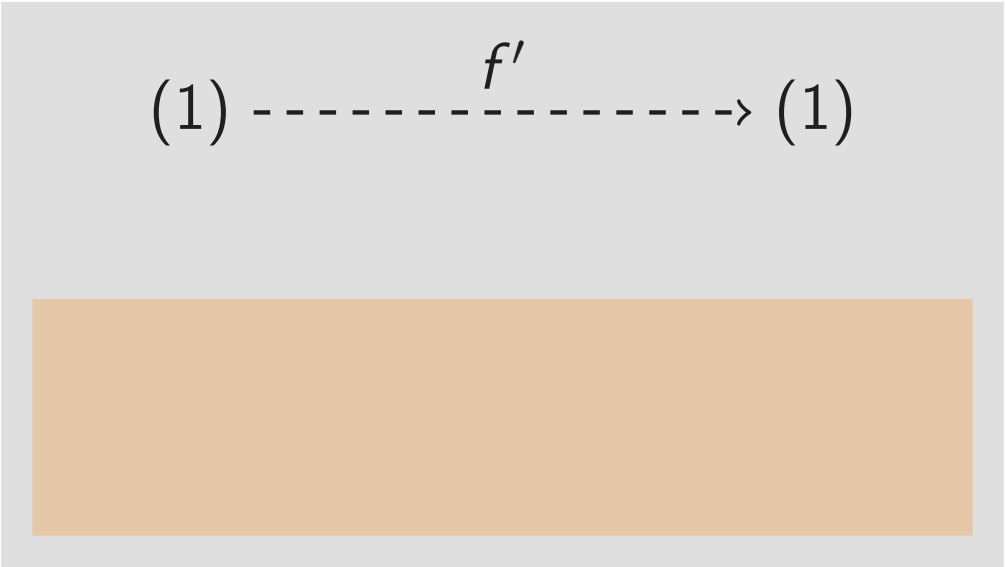

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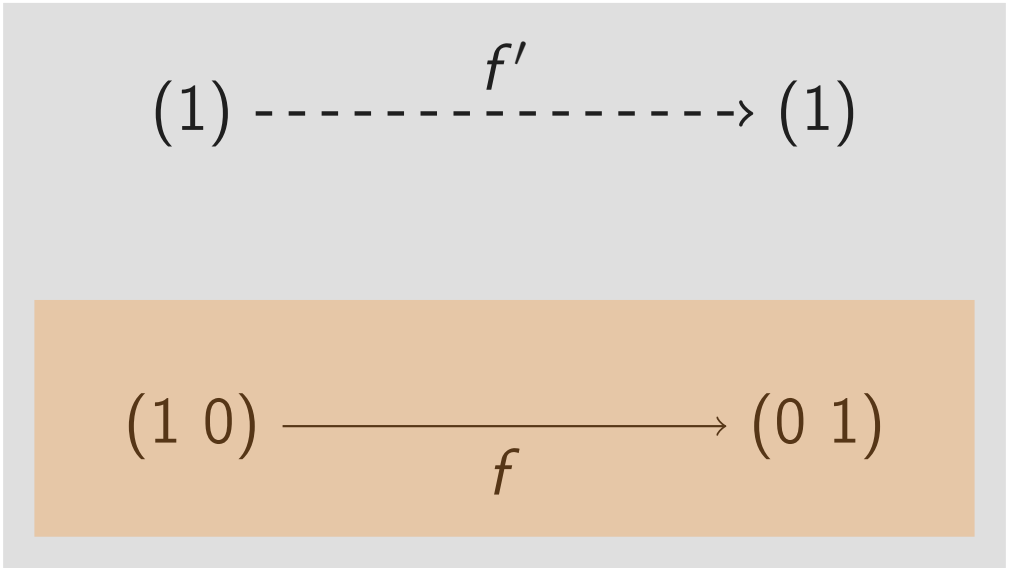

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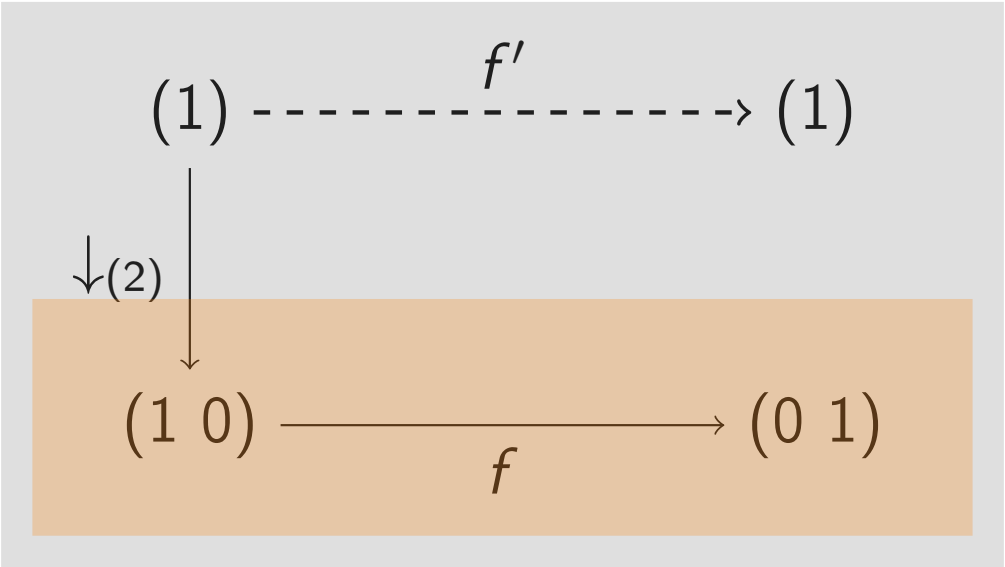

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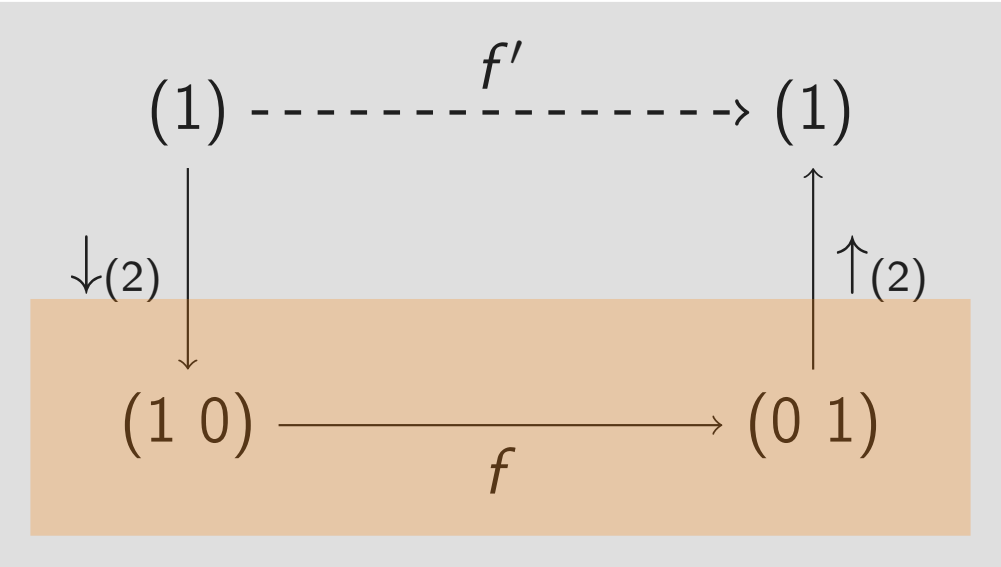

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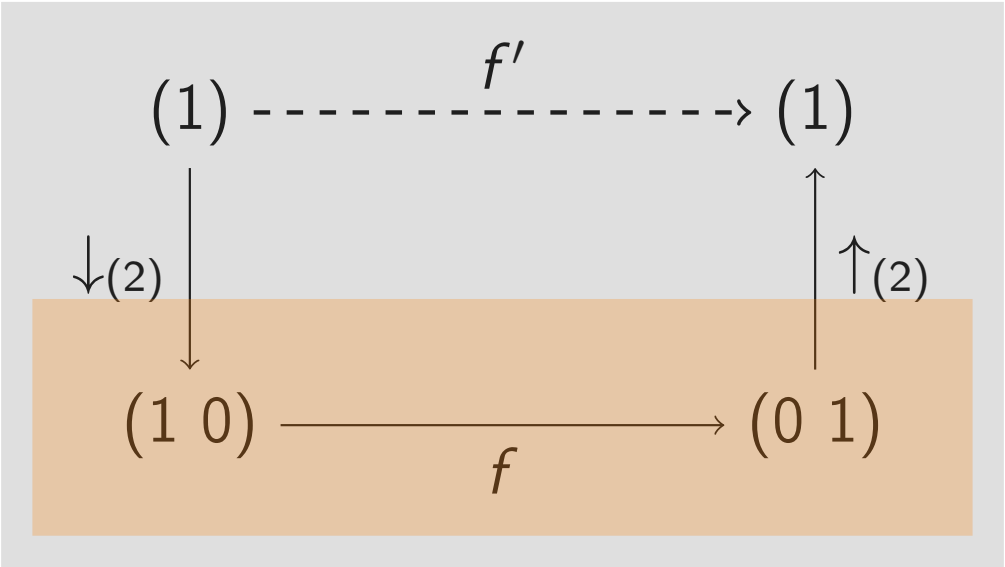

This suggests the reasoning principle

$$\frac{w_1' \downarrow_w w_1 \qquad w_2 \uparrow_w w_2'}{w_1 \multimap w_2 \uparrow_w w_1' \multimap w_2'}$$

More complex principles can be found for $w_1 \multimap w_2 \downarrow_w w_1' \multimap w_2'$

Outline

From Semantics to Syntax

$$
\begin{array}{rcl}
e & ::= & x \\
& | & \lambda x.e \\
& | & e\ e \\
& | & (e, e) \\
& | & \texttt{let}\ (x, x) = e\ \texttt{in}\ e \\
& | & \texttt{fix}\ e \\
& | & c \\
& | & op\ e \\
& | & \texttt{merge}\ p\ e\ e \\
& | & e\ \texttt{when}\ p \\
p & ::= & c^*(c^+)
\end{array}
\qquad
\begin{array}{rcl}
t & ::= & dt :: ct \\
& | & t \otimes t \\
& | & t \multimap t \\
dt & ::= & \mathbf{bool} \mid \mathbf{int} \mid \ldots \\
ct & ::= & p \\
& | & ct\ \mathbf{on}\ ct \\
\\
\\
\\
\Gamma & ::= & \square \\
& | & \Gamma, x : t
\end{array}
$$

Typing Buffers

Sub

$$\frac{\Gamma \vdash e : t \qquad \vdash t <:_k t'}{\Gamma \vdash e : t'}$$

AdaptFun

$$\frac{t_1' <:_{k_1} t_1 \qquad t_2 <:_{k_2} t_2'}{t_1 \multimap t_2 <:_0 t_1' \multimap t_2'}$$

Typing Feedback

$$\text{Fix}\ \frac{\Gamma \vdash e : t \multimap t' \qquad \vdash t' <:_1 t \qquad \vdash t' \text{ value}}{\Gamma \vdash \texttt{fix}\ e : t'}$$

Typing Local Time Scales

$$\text{Scale}\ \frac{\vdash \Gamma \downarrow_{ct} \Gamma' \qquad \Gamma' \vdash e : t' \qquad \vdash t' \uparrow_{ct} t}{\Gamma \vdash e : t}$$

Soundness and Realizability

Two semantics: untyped $\mathcal{K}[\![\_]\!]$ and typed $\mathcal{S}[\![\_]\!]$, look like

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Soundness theorem

The statics (typing) and dynamics (semantics) agree:

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Some interesting, more or less direct corollaries:

- The typed semantics is causal

  $$\forall e, dt, ct, \mathcal{S}[\![\vdash e : dt :: ct]\!] \text{ is total}$$

- *Synchronizing* the clocked semantics gives clocked one

  $$\forall e, t, \mathcal{S}[\![\vdash e : t]\!] = sync_t\ \mathcal{K}[\![\vdash e : t]\!]$$

Soundness proof (1/2)

- First, define the set of *realizers* of some type t:

$$
\begin{array}{lcl}
\mathcal{W}_t & \subseteq & \mathcal{S}[\![t]\!] \\
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- The soundness theorem then becomes a corollary of the *adequacy lemma*: for all Γ, e and t, we have

$$\forall \gamma \in \mathcal{W}_\Gamma, (\mathcal{S}[\![\Gamma \vdash e : t]\!]\ \gamma) \in \mathcal{W}_t$$

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- Unfortunately, it does not work!

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- And restate the adequacy lemma:

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- An essential lemma for fixpoints:

$$\forall t, t', \forall k, n \in \mathbb{N}, \forall xs \in \mathcal{W}_{t}^{n}, (\mathcal{S}[\![\vdash t <:_k t']\!]\ xs) \in \mathcal{W}_{t'}^{n+k}$$

Outline

Related work and Inspiration

- Lustre (Caspi, Halbwachs et al.)
  - General conceptual setting
- Lucid Synchrone (Caspi, Pouzet et al.)
  - Clocks as types
  - Separate compilation
- Lucy-n (Mandel, Plateau, Pouzet)
  - Buffers, adaptability
  - Ultimately periodic clocks
- Clock Domains in ReactiveML (Mandel, Pasteur)
  - Local time scales
- Geometry of Synthesis, Verity (Ghica)
  - Linear HOFs to circuits via **G()** (from Abramsky, Girard)
- Cyclic Scheduling of *DFs (Lee, Munier-Kordon, etc.)
  - Algorithms for type inference with periodic clocks

Conclusion and Perspectives

- A setting for unified clocking / initialization / causality analysis
  - The full type system is not *overly* complex
  - Local time scales important for modularity
  - No need for a scheduling pass after typing
- Relies on standard programming language theory
  - Denotational semantics, Types, Realizability
  - Realizability is a powerful tool. Too powerful?
- Lots of remaining questions
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Thank you!

Bonus Slides

Scatter/Gather: The Whole Story

$$\text{DownArrowBin}\quad \frac{\vdash t_1' \uparrow_{ct} t_1 \qquad \vdash t_2 \downarrow_{ct} t_2' \qquad ct \leq (1)}{\vdash t_1 \multimap t_2 \downarrow_{ct} t_1' \multimap t_2'}$$

$$\text{DownArrowPos}\quad \frac{\vdash t_1' \downarrow_{ct'} t_1 \qquad \vdash t_2 \uparrow_{ct'} t_2' \qquad ct \textbf{ on } ct' = (1)}{\vdash t_1 \multimap t_2 \downarrow_{ct} t_1' \multimap t_2'}$$

$$\text{DownOn}\quad \frac{\vdash t \downarrow_{ct} t'' \qquad \vdash t'' \downarrow_{ct'} t'}{\vdash t \downarrow_{ct \textbf{ on } ct'} t'}$$

Putting it all together (1/2)

Take $f(x, y) = (0.y, x)$. Is the smallest fixpoint of f total? Why?

This problem is equivalent to the scheduling of this Lustre code:

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x = 0 fby y
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Consider the signature below:

$$f \ :: \ (0\,1) \otimes 0(0\,1) \multimap (1\,0) \otimes (0\,1)$$

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It mimics the growth of partial streams in $lfp\ f = \bigsqcup_{i \geq 0}(f^i \bot)$:

x	$f\ x$
$(\bot, \bot)$	$(0.\bot, \bot)$
$(0.\bot, \bot)$	$(0.\bot, 0.\bot)$
$(0.\bot, 0.\bot)$	$(0.0.\bot, 0.\bot)$
$(0.0.\bot, 0.\bot)$	$(0.0.\bot, 0.0.\bot)$
$\ldots$	$\ldots$

Putting it all together (2/2)

So, with $f :: (0\,1) \otimes 0(0\,1) \multimap (1\,0) \otimes (0\,1)$, since

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Interestingly, something happens to the internal buffers

Inside view	Outside view
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