Which simple types have a unique inhabitant?

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- in a given type system (STLC with atoms, products and sums)
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Motivation: a principal way to study **code inference**.

We can infer the instance declaration for the exception monad $A \mapsto A + E$:

$$X + E \to (X \to Y + E) \to Y + E$$

STLC with sums

$$\frac{\Delta, x : A \vdash t : B}{\Delta \vdash \lambda x.\, t : A \to B} \qquad \frac{\Delta \vdash t : A \to B \qquad \Delta \vdash u : A}{\Delta \vdash t\ u : B}$$

$$\frac{\Delta \vdash t : A \qquad \Delta \vdash u : B}{\Delta \vdash (t, u) : A * B} \qquad \frac{\Delta \vdash t : A_1 * A_2}{\Delta \vdash \pi_i\ t : A_i} \qquad \Delta, x : A \vdash x : A$$

$$\frac{\Delta \vdash t : A_i}{\Delta \vdash \sigma_i\ t : A_1 + A_2}$$

$$\frac{\Delta \vdash t : A_1 + A_2 \qquad \Delta, x_1 : A_1 \vdash u_1 : C \qquad \Delta, x_2 : A_2 \vdash u_2 : C}{\Delta \vdash \delta(t,\ x_1.u_1,\ x_2.u_2) : C}$$

$\beta\eta$-equivalence

$$(\lambda x.\, t)\; u \to_\beta t[u/x] \qquad\qquad (t : A \to B) =_\eta \lambda x.\, t\; x$$

$$\pi_i\; (t_1, t_2) \to_\beta t_i \qquad\qquad (t : A * B) =_\eta (\pi_1\; t, \pi_2\; t)$$

$$\delta(\sigma_i\; t,\, x_1.u_1,\, x_2.u_2) \to_\beta u_i[t/x_i]$$

$$\forall C[\square], \quad C[t : A + B] =_\eta \delta(t,\, x.C[\sigma_1\; x],\, x.C[\sigma_2\; x])$$

Equivalence algorithm dedicable — since 1995.

A primer on focusing

Sequent calculus

(Can be done in natural deduction, but less regular)

$$\frac{\Delta \vdash A \qquad \Delta, B \vdash C}{\Delta, A \to B \vdash C} -$$

$$\frac{\Delta, A \vdash B}{\Delta \vdash A \to B}$$

$$\frac{\Delta, A_i \vdash C}{\Delta, A_1 * A_2 \vdash C} -$$

$$\frac{\Delta \vdash A_1 \qquad \Delta \vdash A_2}{\Delta \vdash A_1 * A_2}$$

$$\frac{\Delta, A_1 \vdash C \qquad \Delta, A_2 \vdash C}{\Delta, A_1 + A_2 \vdash C}$$

$$\frac{\Delta \vdash A_i}{\Delta \vdash A_1 + A_2} +$$

Invertible vs. non-invertible rules.
Negatives (interesting on the left): products, arrow, atoms.
Positives (interesting on the right): sums, atoms.

Invertible phase

$$\dfrac{\dfrac{?}{X + Y \vdash X}}{X + Y \vdash X + Y}$$

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Focusing restriction 1: invertible phases

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– and their order does not matter.

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– and their order does not matter.

Imposing this restriction gives a single proof of $(X \to Y) \to (X \to Y)$ instead of two ($\lambda f.\, f$ and $\lambda f.\, \lambda x.\, f\ x$).

Non-invertible phases

After all invertible rules, $\Gamma_n \vdash P_p$

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When a principal formula is selected for non-invertible rule, they should be applied as long as possible – until its polarity changes.

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Focusing restriction 2: non-invertible phase

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Completeness: this restriction preserves provability. **Non-trivial !**
Example of removed redundancy:

$$\cfrac{\cfrac{\cfrac{X_2, \qquad Y_1 \vdash A}{X_2 * X_3, \qquad Y_1 \vdash A}}{X_2 * X_3, \quad Y_1 * Y_2 \vdash A}}{X_1 * X_2 * X_3, Y_1 * Y_2 \vdash A}$$

This was focusing

Focused proofs are structured in alternating phases,
invertible (boring) and non-invertible (focus).

Phases are forced to be as long as possible – to eliminate duplicate proofs.

The idea is independent from the proof system.
Applies to sequent calculus or natural deduction;
intuitionistic, classical, linear, you-name-it logic.

On proof terms, these restrictions correspond to $\beta\eta$-normal forms (at least for products and arrows). But the fun is in the search.

Back to unique inhabitants

You said β-short η-long normal forms?

In presence of negative connectives only (or positive only), **focused** proof search enumerates distinct normal forms.

This fails when sums (positives) are mixed with arrows (negatives).

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(Ad break)

The obvious idea...

Enumerate all derivations in a reasonable (focused) system.
Remove duplicates using the equivalence algorithm.
Stop if two proofs are found.

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$f : (X \to Y + Y), x : X \vdash ? : X$

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$\dots$

We need a more canonical proof search process.

Terminology (a contribution?)

We consider search processes of the form "enumerating the derivations of this restricted system of inference rules".

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Focused proof search quotiented by equivalence: complete, canonical, non-terminating.

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Focused proof search: computationally complete, not canonical.
Focused proof search quotiented by equivalence: complete, canonical, non-terminating.
Forward method with subsumption: provability complete but not unicity complete.

Our approach

We promised an algorithm that decides uniqueness of inhabitation.
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We distinguish **specification** and **implementation**.

Specification: a novel focused logic that is **computationally complete** and **canonical**.

Implementation: a restriction of this logic that is **unicity complete** and **terminating**.

Sum equivalence

```
fun f x ->
  match f x with
    ...
    fun y ->
      match f x with
        ...
        match f y with ...
```

Sum equivalence algorithms move case-splits **up** — then merge them.

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Moving up corresponds to the core idea of **maximal multi-focusing**: non-invertible phases should happen as early as possible.

http://gallium.inria.fr/~scherer/drafts/multifoc_sums.pdf

Backward search for maximal multi-focusing?

Maximality is a global property.

Building a maximal proof by goal-directed proof search seems difficult. At a focusing point $\Gamma \vdash ? : P$, we would have to guess which non-invertible phases will be used **deep** in ?, and perform them now.

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Building a maximal proof by goal-directed proof search seems difficult. At a focusing point $\Gamma \vdash ? : P$, we would have to guess which non-invertible phases will be used **deep** in ?, and perform them now.

Our answer: let's perform **all** the non-invertible sequences we can, even those that won't be needed by any proof.
Saturating proof search: "Cut the positives as soon as possible"

Polarized simply-typed lambda-calculus, with non-biased atoms

$$
\begin{array}{lll}
A, B, C, D & ::= & & \text{types} \\
& \mid X, Y, Z & & \text{atoms} \\
& \mid P, Q & & \text{positive types} \\
& \mid N, M & & \text{negative types} \\
\\
P, Q & ::= A + B & & \text{positive} \\
N, M & ::= A \to B \mid A * B & & \text{negative} \\
\\
P_{\text{at}}, Q_{\text{at}} & ::= P, Q \mid X, Y, Z & & \text{positive or atomic} \\
N_{\text{at}}, M_{\text{at}} & ::= N, M \mid X, Y, Z & & \text{negative or atomic} \\
\\
\Gamma & ::= \texttt{varmap}(N_{\text{at}}) & & \text{negative or atomic context} \\
\Delta & ::= \texttt{varmap}(A) & & \text{general context}
\end{array}
$$

(Note: we could have a positive product as well, it works.)

Focused natural deduction for intuitionistic logic

$$\textsc{inv-sum}\ \frac{\Gamma; \Delta, x : A \vdash_{\mathsf{inv}} t : C \qquad \Gamma; \Delta, x : B \vdash_{\mathsf{inv}} u : C}{\Gamma; \Delta, x : A + B \vdash_{\mathsf{inv}} \delta(x,\ x.t,\ x.u) : C}$$

$$\textsc{inv-arr}\ \frac{\Gamma; \Delta, x : A \vdash_{\mathsf{inv}} t : B}{\Gamma; \Delta \vdash_{\mathsf{inv}} \lambda x.\, t : A \to B}$$

$$\textsc{inv-pair}\ \frac{\Gamma; \Delta \vdash_{\mathsf{inv}} t : A \qquad \Gamma; \Delta \vdash_{\mathsf{inv}} u : B}{\Gamma; \Delta \vdash_{\mathsf{inv}} (t, u) : A * B}$$

$$\textsc{inv-end}\ \frac{\Gamma, \Gamma' \vdash_{\mathsf{foc}} t : P_{\mathsf{at}}}{\Gamma; \Gamma' \vdash_{\mathsf{inv}} t : P_{\mathsf{at}}}$$

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INV-ARR
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INV-PAIR
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FOC-ELIM
$$\frac{\Gamma \vdash n \Downarrow P \qquad \Gamma; x : P \vdash_{\text{inv}} t : Q_{\text{at}}}{\Gamma \vdash_{\text{foc}} \texttt{let}\ x = n\ \texttt{in}\ t : Q_{\text{at}}}$$

FOC-INTRO
$$\frac{\Gamma \vdash t \Uparrow P}{\Gamma \vdash_{\text{foc}} t : P}$$

FOC-ATOM
$$\frac{\Gamma \vdash n \Downarrow X}{\Gamma \vdash_{\text{foc}} n : X}$$

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$$\frac{\Gamma \vdash t \Uparrow A_i}{\Gamma \vdash \sigma_i\ t \Uparrow A_1 + A_2}$$

INTRO-END
$$\frac{\Gamma; \emptyset \vdash_{\text{inv}} t : N_{\text{at}}}{\Gamma \vdash t \Uparrow N_{\text{at}}}$$

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$$\frac{\Gamma \vdash t \Uparrow P}{\Gamma \vdash_{\mathsf{foc}} t : P}\ \textsc{foc-intro}$$

$$\frac{\Gamma \vdash n \Downarrow X}{\Gamma \vdash_{\mathsf{foc}} n : X}\ \textsc{foc-atom}$$

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$$\frac{\Gamma \vdash n \Downarrow A \to B \qquad \Gamma \vdash u \Uparrow A}{\Gamma \vdash n\ u \Downarrow B}\ \textsc{elim-arr}$$

Saturating focused natural deduction

$$\frac{\Gamma; \Delta, x : A \vdash_{\mathsf{sinv}} t : C \qquad \Gamma; \Delta, x : B \vdash_{\mathsf{sinv}} u : C}{\Gamma; \Delta, x : A + B \vdash_{\mathsf{sinv}} \delta(x,\ x.t,\ x.u) : C}\ \textsc{sinv-sum}$$

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$$\textsc{sat-atom}\quad \frac{\Gamma \vdash n \Downarrow X}{\Gamma; \emptyset \vdash_{\mathsf{sat}} n : X}$$

$$\textsc{intro-sum}\quad \frac{\Gamma \vdash t \Uparrow A_i}{\Gamma \vdash \sigma_i\ t \Uparrow A_1 + A_2}$$

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Saturating focused natural deduction

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saturation

$X, (X \to Y + Y) \vdash Y$

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ELIM-PAIR
$$\frac{\Gamma \vdash n \Downarrow A_1 * A_2}{\Gamma \vdash \pi_i\ n \Downarrow A_i}$$

ELIM-START
$$\frac{(x : N_{\mathsf{at}}) \in \Gamma}{\Gamma \vdash x \Downarrow N_{\mathsf{at}}}$$

ELIM-ARR
$$\frac{\Gamma \vdash n \Downarrow A \to B \qquad \Gamma \vdash u \Uparrow A}{\Gamma \vdash n\ u \Downarrow B}$$

Saturating focused natural deduction

$$\textsc{sinv-sum}\quad \frac{\Gamma; \Delta, x : A \vdash_{\mathsf{sinv}} t : C \qquad \Gamma; \Delta, x : B \vdash_{\mathsf{sinv}} u : C}{\Gamma; \Delta, x : A + B \vdash_{\mathsf{sinv}} \delta(x,\ x.t,\ x.u) : C}$$

$$\textsc{sinv-arr}\quad \frac{\Gamma; \Delta, x : A \vdash_{\mathsf{sinv}} t : B}{\Gamma; \Delta \vdash_{\mathsf{sinv}} \lambda x.\, t : A \to B}$$

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$$\textsc{sinv-end}\quad \frac{\Gamma; \Gamma' \vdash_{\mathsf{sat}} t : P_{\mathsf{at}}}{\Gamma; \Gamma' \vdash_{\mathsf{sinv}} t : P_{\mathsf{at}}}$$

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canonicity

$$X, (X \to X + X) \vdash X$$

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finite derivations

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$$\frac{\Gamma \vdash t \Uparrow P}{\Gamma; \emptyset \vdash_{\mathsf{sat}} t : P}$$

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Breaks computational completeness

Termination 1: bounded multisets

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In fact $n := 2$ suffices – for any bound n, you find at least n proofs.

Termination 2: recurring at most twice

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Computational completeness is lost, but
Unicity completeness regained.

Algorithm

Our algorithm searches for all saturated proofs under these two search-space restriction.

Optimization 1: redundancy (elim and intro).
Optimization 2: monotonicity.

(The logic again)

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Future work: extend to polymorphism and dependent types.

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Thanks. Any question?

Paper draft:
gallium.inria.fr/~scherer/drafts/unique_stlc_sums.pdf